H. L. RIETZ, University of Iowa
J. F. RITT, Columbia University
W. E. ROTH, University of Wisconsin
D. A. ROTHROCK, Indiana University
LULU L. RUNGE, University of Nebraska

R. G. SANGER, University of Chicago
HAZEL E. SCHOONMAKER, Hartwick College
E. W. SCHREIBER, State Teachers College, Macomb, Ill.
W. A. SHEWHART, Bell Telephone Laboratories
C. GRACE SHOVER, Connecticut College
E. B. SKINNER, University of Wisconsin
H. E. SLAUGHT, University of Chicago
C. H. SMILEY, Brown University
H. W. SMITH, Oklahoma A and M College
W. M. SMITH, Lafayette College
I. S. SOKOLNIKOFF, University of Wisconsin
VIVIAN E. SPENCER, University of Pittsburgh
ANNA A. STAFFORD, University of Chicago
MARION E. STARK, Wellesley College
R. C. STEPHENS, Knox College
GUY STEVENSON, University of Louisville
ELLEN C. STOKES, New York State College for Teachers
E. B. STOUFFER, University of Kansas
C. J. STOWELL, McKendree College

MILDRED E. TAYLOR, Mary Baldwin College
V. B. TEACH, Armour Institute of Technology
MRS. H. L. TITSWORTH, Sophie Newcomb College
BIRD M. TURNER, West Virginia University

E. B. VAN VLECK, University of Wisconsin

R. J. WALKER, Princeton University
MARIE J. WEISS, Sophie Newcomb College
ELLEN E. WILEY, Middlebury College
W. L. WILLIAMS, University of South Carolina
E. B. WILSON, Harvard School of Public Health
ELIZABETH W. WILSON, Cambridge, Mass
MARGARETE C. WOLF, University of Wisconsin
F. E. WOOD, Northwestern University
META A. WOOD, Lincoln School, New York, N. Y.

B. F. YANNEY, College of Wooster
C. H. YEATON, Oberlin College
E. I. YOWELL, University of Cincinnati

The Science Congress, sponsored jointly by the American Association for the Advancement of Science and A Century of Progress, was held from June 19 to June 30, the first week being devoted to pure science and the second to applied science. The notable feature of the first week, within which the mathematics meetings were held, was the presence of twenty-seven distinguished foreign men of science as guests of A Century of Progress and the American Association. It was the great pleasure of the mathematics group to entertain Professor Lipót Fejér of the University of Budapest, Professor Niels Bohr of the University of Copenhagen, and Professor Tullio Levi-Civita of the University of Rome, as well as Professor Enrico Bompiani of the University of Rome who was associated with the science exhibits of the Italian Government. A reception was given to the foreign guests on the evening of June 19 in the Hall of Science at which welcoming addresses were made by Rufus Dawes, president of A Century of Progress, Professor A. H. Compton of the Local Committee, Professor J. J. Abel, president of the American Association for 1932, and Professor Henry Crew who was in charge of science for the Exposition; and a response was made by Professor Bohr in behalf of the foreign guests. These guests formed the nucleus for a number of symposia held during the week in connection with the meetings of the various science organizations. A gala dinner was given on Thursday evening at the Hotel Stevens in honor of the foreign guests.

Numerous mathematical exhibits were shown in the northern part of the Hall of Science. Here, in the center of an octagonal room, were displayed a set of slides, illustrating the history of mathematics, by Professor L. C. Karpinski of

the University of Michigan, shown by means of four projection lanterns, one for each of the four great fields into which mathematics is divided. Near this were shown striking mathematical surfaces made by Mr. C. E. Johansson, and elaborate geometrical models by Doctor Saul Pollock. Nearby were seen Michelson's harmonic analyzer and mathematical instruments used in the calculations involved in his notable scientific investigations. Here also were to be found the Galton Quincunx in which probability curves are formed by steel balls deviated in their falling by steel pegs in "penny-slot-machine" fashion, another exhibit in which the probability of a rod falling on any one of a group of parallel lines is used to determine experimentally the value of π, a machine for the computation of simple harmonic motions, numerous geometrical and kinematical devices, and machines in the nature of mathematical recreations. In this same section was the number-theory machine of Doctor D. H. Lehmer which was described by him on the program of the Mathematical Association. Neighboring halls and corridors exhibited in popular form many of the striking attainments in physics, chemistry and the other sciences. A notable exhibit in the Hall of Social Sciences was the collection of early American arithmetics and other text-books from the well-known collection of Mr. George A. Plimpton of New York City.

The American Mathematical Society held sessions throughout the week with the largest summer attendance in its history. On Tuesday afternoon a joint session of the Society and Section A of the American Association, with the American Physical Society as their guests, was addressed by Professor Tullio Levi-Civita on "Some mathematical aspects of the new mechanics," and by Professor G. D. Birkhoff on "Quantum mechanics and asymptotic series." A second symposium of the Society and Section A on Wednesday morning was addressed by Professor Lipót Fejér on "The infinite sequences arising in the theories of harmonic analysis, of interpolation, and of mechanical quadratures," by Professor C. N. Moore on "The use of Cesàro means in determining criteria for Fourier's constants," and by Professor Dunham Jackson on "Certain problems of closest approximation." A symposium on geometry was held on Wednesday afternoon at the Italian Building on the Exposition grounds; addresses were delivered by Professor Levi-Civita on "Nets on a surface and extension of trigonometry," by Professor W. C. Graustein on "Invariant methods in differential geometry," and by Professor Enrico Bompiani on "Deformations of higher species of surfaces and manifolds." The Society also held a session devoted to number theory on Friday afternoon at which Professor L. E. Dickson spoke on "Recent progress in additive number theory," followed by a number of contributed papers. Sessions were held on Monday afternoon and Thursday and Friday mornings for the reading of short contributed papers.

Mathematicians and their guests were housed in very comfortable quarters in Judson Court on the south side of the Midway, opposite the University campus; spacious lounges and social rooms were available for their use. The Common Room at Eckhart Hall proved to be very convenient for social purposes,

tea being served several afternoons at the close of the sessions by the ladies of the faculty. There was a certain loss of time in the wide separation in the University campus and the Exposition grounds, but with the convenience of transportation this was more than offset by the freedom which the mathematicians had to choose their own personal combination of meetings and pleasure trips. The good services of the committee on local arrangements were recognized in a suitable vote by the Trustees including the recognition of the extensive plans made in advance by Professor Everett, the chairman of the committee.

About two hundred fifty members and guests attended the joint dinner of the mathematical organizations Friday evening at Judson Court. After the delightful dinner had been served, Professor Bliss as toastmaster introduced the various speakers. Professor Fejér expressed his appreciation of the cordial welcome with which he had been received. At the request of Professor Bliss, he delivered this in his native language and this was followed by a translation ably given by Mr. Medgycy, the Hungarian Consul-General at Chicago. Professor Levi-Civita then expressed his pleasure in being the guest of A Century of Progress and his good fortune in being associated with American mathematicians. His speech, again at Professor Bliss's request, was delivered in Italian and was translated by Professor Bompiani. Professor Bohr spoke of the present status of university professors in Europe, particularly those in mathematics and physics. Professors Bompiani, Coble and Dresden spoke more briefly, the latter two representing the Society and Association. All these addresses, together with the jovial introductions by Professor Bliss, constituted a delightful evening.

The program of the Mathematical Association consisted of a joint session with Section A on Tuesday morning at Eckhart Hall, University of Chicago, and a second joint session with Section A Thursday afternoon in South Hall of the Hall of Science on the Exposition grounds, and a joint session with Section A and the American Mathematical Society Saturday afternoon at Northwestern University, Evanston. President Dresden presided at the first two sessions and Professor Slaught at the third. The Trustees recognized by a resolution the very effective organization of a good program of speakers, as prepared by the program committee which consisted of Professors Louis Brand, Mayme I. Logsdon, E. R. Smith, and E. J. Moulton, chairman. The program follows, together with abstracts of some of the papers numbered in accordance with their place on the program.

First Joint Session of the Association with Section A of the American Association

1. "The lag of mathematics behind literature and art in the early centuries" by Professor H. E. Slaught, University of Chicago.

2. "Mathematics and art" by Professor G. D. Birkhoff, Harvard University.

3. "The postulational method in mathematics" by Professor E. V. HUNTINGTON, Harvard University.

1. The address by Professor Slaught will appear in an early issue of the MONTHLY.

2. The possibility of a connection between mathematics and art was considered by Professor Birkhoff from the point of view of the theory of aesthetic measure, first presented to the Association at Ithaca in 1925. In its applications to polygonal forms, ornaments and tilings, vases, harmony, melody, and the musical quality in poetry, certain elements of order in the aesthetic object are taken to be of fundamental importance. According to his theory the enjoyability of such an object depends primarily on the density of these elements of order. It is an interesting fact that these elements correspond in general to simple mathematical relations, and this suggests that aesthetic enjoyment may arise in part from the intuitive appreciation of such relations. For example, primitive man doubtless enjoyed the beauty of form of the moon or sun because of an intuitive appreciation of the geometric relation of circularity. In the applications considered, the most important relations of this kind were found to involve geometry, the theory of groups and of periodic functions. It was only in this special sense that a connection was made out between mathematics and art, despite the fact that many writers from the time of antiquity have endeavored to establish a mystical connection between the two fields.

3. This paper will appear in an early issue of the MONTHLY. It outlines three "levels of classification" of a given universe: (1) primary observations; (2) secondary observations, which are generalized into "hypothetical laws"; and (3) interrelations among these hypothetical laws. It is only on this third level that "mathematical propositions" arise. Thus, a set of n laws divides the given universe into 2^n compartments, and the assertion that one of these compartments is "empty" is what is meant by a mathematical proposition.

SECOND JOINT SESSION OF THE ASSOCIATION WITH SECTION A OF THE AMERICAN ASSOCIATION

1. "Fundamental concepts in the theory of probability" by Doctor T. C. FRY, Bell Telephone Laboratories.

2. "Applications of mathematics to real estate problems" by Doctor H. A. BABCOCK, Evanston, Illinois.

3. "A number theory machine" by Doctor D. H. LEHMER, Altadena, California.

1. Doctor Fry presented in a very concrete fashion the simpler concepts in the theory of probability and led up through numerous illustrative examples to exact definitions of "probability" and related terms. His paper will appear in full in an early issue of the MONTHLY.

2. In the valuation of land and the improvements thereon to determine what price a prospective purchaser is warranted in paying and the prospective seller

is warranted in accepting for any specific property, the fair price may be estimated, in the case of income producing properties, by estimating the future series of net receipts from the property and computing the present, or initial, worth of the series at an interest rate which reflects the uncertainties involved in the prediction of future events. Also, in the case of an aggregate of parcels of land to be sold over a period of time to those who wish to utilize the parcels the series of net returns from future sales may be estimated and the initial value of the series computed. The initial value of a series of net receipts $r_1, r_2, \cdots, r_n, \cdots r_N$ may be written

$$V_0 = \sum_1^N r_n v^n, \qquad v = 1/(1+i).$$

Doctor Babcock applied the principles given in this MONTHLY for March 1933 to the real estate problems appearing in a community in which the population is increasing and hence bringing about demands for additional land to serve as sites for additional buildings. The difference in prices paid for land arises from the fact that certain sites are more desirable for the intended use than others, the type of intended use being a factor in determining the maximum price. Moreover the owner with land to sell will hold land in some cases for many years, refusing offers to buy at low prices and waiting for a demand at a higher price. Assume an aggregate of land with a total area A, with a single ownership, assume that there are annual demands for various parts of this land by prospective purchasers and therefore at various unit prices, the annual demand in each price group being the same in each year. The owner of area A thus has an opportunity to sell over a period of time, parcel by parcel. The initial value of the area, subject to tax on the unsold portions each year, if the demand in one price group only is accepted, will be

$$V_0 = R_1 w_1 a_{\overline{N}|\, i+\rho},$$

where R_1 is the demand rate, w_1 the unit price, N the area A divided by the rate R_1, i the rate of return, ρ the tax rate and $a_{\overline{N}|}$ the annuity factor at the rate $(i+\rho)$ for N years.

The question arises whether the owner would not produce a greater initial value for himself by selling some of his land at a lower price provided he disposes of it faster. If the various unit prices are designated by $w_1, w_2, \cdots, w_s$ and the corresponding demand rates by $R_1, R_2, \cdots, R_s$, and if the number of years that he will accept these demands are designated by $N_1, N_2, \cdots, N_s$, the initial value of the area will be

$$V_0 = R_1 w_1 a_{\overline{N_1}|} + R_2 w_2 a_{\overline{N_2}|} + \cdots + R_s w_s a_{\overline{N_s}|},$$

subject to the condition that

$$R_1 N_1 + R_2 N_2 + \cdots + R_s N_s = A.$$

The condition that V_0 be a maximum gives the values of the several N's and the amount of area to be sold at each price.

Under certain circumstances some of the partial areas thus determined will be negative, which is physically inadmissible. It is found, however, that if the price groups are arranged in descending order, all the negative areas appear below a certain price group, which the author has called the critical price, and that the value of the area produced by accepting just those demands which occur at prices ranging down to the critical price, is greater than any other combination which does not involve negative areas, regardless of the relative sizes of the demand rates. It appears also that the length of time the demand at any price will be accepted is less the lower the price. If we assume that the rejected demands are filled somewhere outside the area A and that the prospective purchasers in all price groups accept land in an area B only because they cannot secure land in A at a sufficiently low price, the important result is found that the distribution of building development in the two areas will be the same when the owners of A and B are in competition as when the total area $(A+B)$ is in one ownership. Similarly for a third area C, and so on.

The speaker derived various consequences from this theory, illustrating it by the instance of Los Angeles where, under the belief that the population would continue to increase at a rapid rate, the building development had been pushed out to a great extent and a large amount of vacant land left behind, and of Baltimore, a less rapidly growing city, where the building development is more compact and there is a smaller percentage of vacant land left behind. He described his method as a start toward a more comprehensive theory which will take account of variations in these demand rates, and indicated paths for development on the practical and theoretical side.

3. The essentials of the explanation of Doctor Lehmer's very interesting number-theory machine have appeared in this Monthly, for August-September 1933. At the instance of A Century of Progress, Doctor Lehmer spent the summer at Chicago and showed each day the workings of his machine as a part of the mathematical exhibit in the Hall of Science.

Following this, Professor L. C. Karpinski of the University of Michigan showed a considerable portion of the slides in the history of mathematics which were being exhibited in the mathematical section of the Hall of Science. He commented briefly on these, pointing out the features of mathematical interest.

Joint Session of the Association with Section A of the American Association and the American Mathematical Society

On Saturday afternoon more than one hundred of the mathematicians went by automobile on an excursion along the North Shore and returned to Harris Hall of Northwestern University, where an address on "Mathematical reminiscences of the World's Fair of 1893" was given by Professor T. F. Holgate of Northwestern University.

Various congresses were organized at the time of the World's Fair under the auspices of the World's Columbian Exposition Auxiliary. The strength of the

mathematical congress, he said, arose from the action of the German government which had an elaborate exhibit of German mathematical books and apparatus, and from the appointment of Professor Felix Klein by the German government as their representative. Professor Klein organized a program of papers by eminent European mathematicians describing the status of mathematical study at that time. About forty papers were given on the program, thirteen or fourteen being by American mathematicians. The Congress met in what is now the Art Institute of Chicago. In connection with this, a smaller colloquium was held at Evanston, these papers comprising the first of the series of colloquium publications of the Mathematical Society. Professor Holgate showed in the form of slides two photographs of the members of the Congress and of the colloquium. As the printed reports of these meetings show, these included numerous well-known mathematicians and mathematical physicists.

After the lecture the ladies of the department of mathematics of Northwestern University served an elaborate tea at one of the attractive dormitories of the University. A resolution of appreciation of the hospitality of the friends at Northwestern University was adopted by rising vote at the afternoon session.

Meeting of the Board of Trustees

Eleven members of the Board of Trustees were present at the meeting at Judson Court on Friday afternoon.

The following thirty-two applicants for membership were elected:

Emily E. Calkins, A.B. (Columbia; William and Mary) Instr., Coll. of William and Mary, Williamsburg, Va.

Rev. Arthur Danzl, A.M. (Columbia) Instr., St. John's Univ., Collegeville, Minn.

Rachel Davison, A.M. (Oberlin) Instr., Houghton Coll., Houghton, N. Y.

L. J. Deck, A.M. (Pennsylvania) Prof., Muhlenberg Coll., Allentown, Pa.

F. A. Downing, Freight Rate Clerk, Atlantic Coast Line R.R., Wilmington, N. C.

D. C. Duncan, Ph.D. (California) Instr., Univ. of California, Berkeley, Calif.

E. W. Franz, A.M. (California) Instr., Junior Coll., Ventura, Calif.

T. N. E. Greville, Ph.D. (Michigan) Highlands, N. C.

W. N. Hallett, Ph.D. (Pennsylvania) Prof., Cedar Crest Coll., Allentown, Pa.

G. M. Hayes, Ph.D. (Fordham) Asst. Prof., Coll. of the City of New York, New York, N. Y.

J. J. Hayes, B.S. (Utah) Instr., Univ. of Utah, Salt Lake City, Utah

E. Marie Hove, M.S. (Iowa) Instr., State Teachers Coll., Wayne, Nebr.

G. J. Kalcik. Senior, St. Nobert Coll., West DePere, Wis.

S. H. Kimball, Ph.D. (Harvard) Instr., Univ. of Rochester, Rochester, N. Y.

M. S. Kovalenko, Ph.D. (Princeton) Asst. Prof., Math. and Astr., Swarthmore Coll., Swarthmore, Pa.

C. H. Lady, A.M. (Southern California) Prof., State Teachers Coll., Slippery Rock, Pa.

T. J. Love, A.M. (Woodstock) Prof., Physics and Math., Loyola Coll., Baltimore, Md.

L. E. Loveridge, Ph.D. (California) Prof., Seton Hall Coll., South Orange, N. J.

R. E. Norris, A.M. (Illinois) Head of Dept., State Teachers Coll., Milwaukee, Wis.

Mabel I. Nowlan, M.S. (Michigan) Head of Dept., Bethel Woman's Coll., Hopkinsville, Ky.

A. S. Peters, M.S. (New York Univ.) Instr., New York Univ., New York, N. Y.

Gladys M. Quigg, A.M. (Penna. State Coll.) Instr., Pennsylvania State Coll., State College, Pa.

C. B. Read, A.M. (Princeton) Asst. Prof., Univ. of Wichita, Wichita, Kans.

W. H. ROBINSON, A.M. (Boston) Head of Dept., Brick Jr. Coll., Bricks, N. C.
L. M. SKOFIELD. Reading, Mass.
C. E. SMITH, A.M. (Swarthmore) Instr., Math. and Astr., State Teachers Coll., Fresno, Calif.
E. R. STABLER, Ed.M. (Harvard) Cambridge, Mass.
M. L. VEST, M.S. (West Virginia) Instr., West Virginia Univ., Morgantown, W. Va.
K. J. WAIDER, A.B. (San Francisco) Instr., Math. and Physics, Univ. of San Francisco, San Francisco, Calif.
C. L. WEAVER, Senior, Kent State Coll., Kent, Ohio
A. MARIE WHELAN, Ph.D. (Johns Hopkins) Asst. Prof., Hunter Coll., New York, N. Y.
MARGARETE C. WOLF, A. B. (Wisconsin) Grad. Student, Univ. of Wisconsin, Madison, Wis.

The Secretary-Treasurer made a report to the Trustees by letter on June 7 and at this meeting concerning the status of part of the funds of the Association which had been tied up in the Oberlin Savings Bank since the "bank holiday" in February. The bonded investments of this bank, as of other banks and investors, have shrunk during the past three years but, to speak briefly, it is expected that this bank will soon resume its activities, make 70% of the frozen deposits available for legitimate business or personal needs of depositors, and issue participation certificates for the remaining 30%, the assets covered by these certificates to remain in the hands of three trustees in the hope that as they become more fluid these participation certificates may be redeemed. By mail vote in June the Trustees authorized the Finance Committee to agree to the plan of reorganization and to accept the participation certificates.[1] The Finance Committee regret that any of the Association funds should be in any way tied up or have an uncertain status, but it seemed to be agreed in the Trustees' discussion that our investments and deposits are widely scattered and that the Association is faring better than many other business organizations. Other than this there has been no default in the income from investments except in one $1000 land trust certificate, quarterly dividends of $13.75 having been passed for the past four quarters.

The Trustees voted to designate The Northern Trust Company of Chicago as one of the depositaries of the Association. The sum of $500.00 recently received on account of the Chace Fund has been deposited there in the form of a certificate of deposit.

Professor Slaught gave an oral report concerning the sets of the Chace Papyrus, a limited number of which are still available. He also reported concerning the sales of the Carus Monographs and the plans of the committee for the future; an announcement will be made to the Association members as soon as more definite plans for the publication of the fifth monograph are formed.

Suggestions as to changes in the conditions of the award of the Chauvenet Prize having been made earlier in the year, a committee on the rules for this prize was appointed in February by President Dresden, as follows: W. B. Ford, chairman; E. T. Bell, R. D. Carmichael, H. E. Slaught, and J. L. Walsh. At the

[1] The bank resumed full activities at the end of August, and a distribution of part of the frozen funds is expected soon.

Chicago meeting a partial report was sent by the chairman, and the Trustees as a committee of the whole adopted a number of tentative conditions and referred these to the committee as suggestions in formulating a new statement of rules.

Growing out of the request of Professor Walter W. Hart that the Association appoint "a strong committee which shall be charged with the business of providing desirable publicity for continued teaching of mathematics in the schools and maintenance of mathematics as a desirable and necessary part of college entrance requirements," there was a general discussion of this question including a further statement by Professor Hart, who was present by invitation. As an outcome of the discussion of this and related questions, it was voted to empower the President to appoint (1) a commission to study the place of mathematics in schools and junior colleges, and (2) a commission to study the training and utilization of advanced students of mathematics. A nucleus of this committee will initially act as a committee of organization and planning and it is expected that fuller information will be given at an early date. In this connection the following resolution was adopted:

> Resolved, That this body would welcome a decision by the National Council of Teachers of Mathematics to hold a meeting at the time of the Boston and Cambridge meetings next December, in affiliation with the A.A.A.S. and the Mathematical Association, so as to come into touch with the members of these associations and with the wider scientific interests.

The trustees authorized the Secretary to issue permits to members of the Association to wear an official pin or button adopted as the official emblem of the Association.

W. D. CAIRNS, *Secretary-Treasurer*

AIR RESISTANCE TO FALLING SPHERES

By E. V. HUNTINGTON, Harvard University

Introduction

Let us consider a sphere, of weight W and radius r, falling from rest in air.

If the resistance of the air is assumed to be proportional to the square of the velocity[1] and also to the area of the cross-section of the sphere,—that is, if the law of resistance is assumed to be

$$R = kAv^2, \tag{1}$$

where R = the retarding force, $A = \pi r^2$, and v = the velocity,—then, as is well known, the distance x fallen in any time t is given by the equation

$$x = \frac{b^2}{g} \log_e \cosh\left(\frac{gt}{b}\right), \tag{2}$$

[1] According to *Encyklopädie der Mathematischen Wissenschaften*, IV, 17 (S. Finsterwalder), p. 161, the quadratic law is generally accepted for velocities from 0.2 to 240 m/sec.

where

$$b = \sqrt{W/(kA)} = \text{the "limiting velocity."}$$

In regard to the constant of proportionality, k, the only experimental determination that applies to the special case of the sphere is, as far as I know, that published by E. H. Hall[1] in 1910. Professor Hall dropped a bronze sphere of diameter 1 inch and weight 73.8 grams from a height of 22.85 meters in a locality where $g = 980.4$ cm/sec^2. By observing the time of fall, 2.176 seconds, and solving by trial and error[2] an equation equivalent to (2), he obtained his result, which may be expressed as

$$k = 0.00058 \text{ lb sec}^2/\text{ft}^4 = 31 \text{ gram sec}^2/\text{m}^4.$$

His experiment is not easy to repeat before a class, however, since it requires the use of an elaborate electrical apparatus for measuring the time of fall correct to one-thousandth of a second.

The purpose of the present paper is to show how the factor k may be determined with similar accuracy by a much simpler experiment *which does not involve the measurement of time.*

The proposed experiment

For this purpose, we use *two* spheres, with weights W_1 and W_2, and cross-sections A_1 and A_2, and *drop them simultaneously.* If $x_1 =$ the distance fallen by the first sphere in any time t, and $x_2 =$ the distance fallen by the second sphere in the same time t, then by (2),

$$x_1 = \frac{b_1^2}{g} \log_e \cosh\left(\frac{gt}{b_1}\right), \tag{3}$$

where $b_1 = \sqrt{W_1/(kA_1)}$, and

$$x_2 = \frac{b_2^2}{g} \log_e \cosh\left(\frac{gt}{b_2}\right), \tag{4}$$

where $b_2 = \sqrt{W_2/(kA_2)}$.

It is easy to adjust the relative sizes and weights so that in a fall of only 50 ft. one of the spheres, say W_2, will be three or four feet ahead of the other sphere when it strikes the ground, and the two distances x_1 and x_2 can be measured with a simple steel tape. (The exact position of the slower sphere, W_1, at the instant when the faster one, W_2, strikes the ground can be found to a sur-

---

[1] *Air resistance to falling inch spheres,* Proceedings of the American Academy of Arts and Sciences, vol. 45, pp. 379–384.

[2] The solution by trial and error can be avoided by using the following expansion:

$$k = \frac{6W}{Ag^2t^2}\left[\frac{x_0 - x}{x_0} + \frac{8}{5}\left(\frac{x_0 - x}{x_0}\right)^2 + \cdots\right],$$

where $x_0 = gt^2/2$, and $x =$ the actual distance fallen in time t.

prising degree of accuracy by the use of two paper "screens," one fixed near the ground and the other adjustable as to height above the ground; simultaneity of sound indicating correct position.)

These two distances, x_1 and x_2, are the only measurements that have to be made. For t can be eliminated between equations (3) and (4), and the resulting equation will contain only the single unknown quantity k.

The resulting expression for k

In order to express the resulting equation for k in the simplest form, and to avoid all confusion in regard to the "units" employed, we introduce the following abbreviations:

$$\alpha = \frac{W_1}{W_2}\frac{A_2}{A_1}, \tag{5}$$

$$\beta = \frac{x_2}{x_1} \tag{6}$$

and

$$u = \frac{kgA_1x_1}{W_1}. \tag{7}$$

(Here α is a pure number, determined by the physical dimensions of the spheres; β is a pure number determined by the observed distances x_1 and x_2; while u is a pure number involving the unknown quantity k).

The equation obtained by eliminating t from (3) and (4) then becomes

$$\sqrt{\alpha}\cosh^{-1}(e^u) = \cosh^{-1}(e^{\alpha\beta u}), \tag{8}$$

in which α and β are known by (5) and (6), and u is the only unknown; and when this equation (8) has been solved for u by trial and error (or better, by series, as explained below), then k itself can be found by

$$k = \frac{W_1}{gA_1x_1}u. \tag{9}$$

Since u is a pure number, the "dimensions" of k will be the same as the "dimensions" of $W_1/(gA_1x_1)$.

Computation by series

The numerical computation involved in the solution of (8) can be greatly simplified by the use of series whenever, as is usually the case, α and u are small numbers.

By the aid of the following expansions (which are easily obtained by elementary calculus methods):

$$\log_e \cosh x = \frac{1}{2}x^2 - \frac{1}{12}x^4 + \frac{1}{45}x^6 - \cdots$$

and

$$\cosh^{-1}(e^x) = \sqrt{2x}\left(1 + \frac{1}{6}x + \frac{1}{120}x^2 + \cdots\right),$$

equation (8), which may be written in the form

$$\beta = \frac{1}{\alpha u} \log_e \cosh \left[\sqrt{\alpha} \cosh^{-1}(e^u)\right], \tag{8a}$$

becomes

$$\beta = 1 + \frac{1}{3}(1 - \alpha)u + \frac{2}{45}(1 - 5\alpha + 4\alpha^2)u^2 + \cdots. \tag{10}$$

Hence a first approximation to u is

$$u_0 = \frac{3(\beta - 1)}{1 - \alpha} + \cdots, \tag{11}$$

and a second approximation is

$$u = u_0 - \frac{2}{15}(1 - 4\alpha)u_0^2 + \cdots; \tag{12}$$

where α and β are known constants by (5) and (6).

The required value of k is then obtained by substituting (11) or (12) in equation (9).

A numerical illustration

In an actual experiment performed in class a few years ago, I used two 6-inch hollow spheres (sold as "floats"), one empty and the other filled with lead, so that $W_1 = 0.60$ lb and $W_2 = 45.75$ lb, while $A_1 = A_2 = 0.20$ ft^2. By dropping the spheres side by side from a tipping shelf outside my class-room window we found, by rather rough measurements (not using "screens"), the values $x_1 = 41.0$ ft. and $x_2 = 45.0$ ft. Hence

$$\alpha = \frac{0.60}{45.75} = 0.0131 \quad \text{and} \quad \beta = \frac{45.0}{41.0} = 1.098,$$

so that equation (8) became

$$0.114 \cosh^{-1}(e^u) = \cosh^{-1}(e^{0.0144u}).$$

The solution of this equation gave $u = 0.29$, whence by direct substitution in (9), we found

$$k = 0.00066 \text{ lb sec}^2/\text{ft}^4 = 35 \text{ gm sec}^2/\text{m}^4.$$

Considering the rough manner in which x_1 and x_2 were measured, the agree-

ment with Professor Hall's earlier result was as close as could be expected.

By repeating the experiment with the use of "screens," and by varying the sizes and weights of the spheres, a very satisfactory test of the range of validity of the law $R=kAv^2$ for spheres could be obtained with very little effort.

The required precision of measurement

Finally, in order to estimate, roughly, the error produced in k by a small error incurred in x_2-x_1 (assuming that the fixed distance x_2 is accurately known), we note that (11) and (9) give as a first approximation

$$k = \frac{3W_1}{gA_1(1-\alpha)x_2^2}(x_2-x_1)\left[1+\frac{2(x_2-x_1)}{x_2}+\cdots\right], \tag{13}$$

whence, by logarithmic differentiation,

$$\frac{dk}{k} = \left[1+\frac{2(x_2-x_1)}{x_2}\right]\frac{d(x_2-x_1)}{x_2-x_1}+\cdots. \tag{14}$$

In the numerical case described above, we have

$$\frac{dk}{k} = [1.2]\frac{d(x_2-x_1)}{x_2-x_1},$$

so that in this case the relative error in k was about 1.2 times the relative error in x_2-x_1.

GEOMETRY AS AN AVOCATION[1]

By A. J. KEMPNER, University of Colorado

It is a truism to say that specialization in mathematics, as in all sciences, has been carried to such an extreme that grave problems are created and great values are endangered. We have algebraists, we have analysts, we have geometers! One asks oneself whether we have mathematicians.

I shall not venture to discuss these questions, although there is food for serious thought in the reflection that large fields in geometry are completely algebraic and analytic in their methods of attack, and that the treatment of important branches of algebra is made possible only by the use of the most advanced and most abstract theories of pure analysis. Indeed, one of the most hopeful signs is that, barring certain narrow fields in each discipline, the *methods* and *tools* are growing more and more the same in all branches; it is the nature of the *problems* which differs widely, rather than the means of solving them.

However this may be, every situation has the advantages of its disadvantages, and I shall avail myself of one of these advantages.

---

[1] Address, delivered by invitation, at the meeting of the Rocky Mountain Section of the Mathematical Association of America, Boulder, Colorado, April 17–18, 1931.

If one is labeled an Algebraist, and ventures to talk about geometry to a Geometer, one is apt to encounter a species of pitying condescension—probably quite justifiably so. However much this attitude may hurt one's pride, it has its compensations; the standard of judgement is decidedly more lenient than toward a fellow geometer, and if one makes a particularly foolish mistake by asserting an incorrect theorem, or by proclaiming as new something thoroughly familiar to the guild, one is, figuratively speaking, good-naturedly patted on the back and told to go back to one's own field.

May I speak, in a sketchy fashion, of some of the geometrical problems which have, at one time or another, interested me? Some of these are of the greatest simplicity, some still seem to me deserving of a little interest.

I. My first remarks deal with the most elementary topics.

From my high school days, I seem to recall only three sparks of intelligence;

I discovered that such a number as e exists, from problems on compound interest.

I found essentially Fermat's theorem $a^{p-1} \equiv 1 \bmod p$, $(a, p) = 1$.

Finally, in trying to understand Newton's law of gravitation, I noticed a little theorem which I have not seen anywhere:

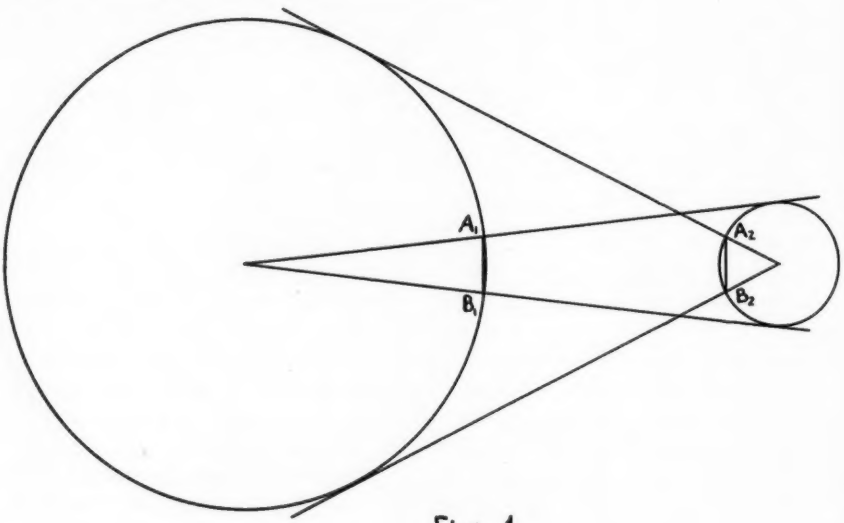


Fig. 1

Always chord $A_1B_1 =$ *chord* A_2B_2.

Later, as a student, in studying conformal mapping I found the following:

Given any two similar figures in the plane, with the same sense of direction. If all segments joining corresponding points are divided in the same ratio, *the points of division again form a figure similar to the given figures.*

The proof of this theorem may be made very simple or fairly complicated, according to the tools employed.[1]

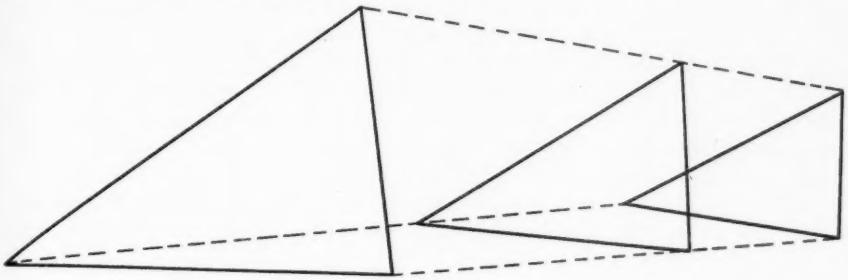


Fig. 2

II. In connection with incidental reading, I mention three problems which have, at various times, arrested my attention:

(*a*). In an old mathematical journal, *The Analyst.* I found this:

Given any curve and a point inside. The chord passing through the point which cuts off the smallest area is bisected at the point.

The "proof" given is indirect and obvious; if the chord cutting off a minimum area were not bisected at P, a contradiction would at once be established by considering another chord through P making a small angle $\Delta\theta$ with the first, and by comparing the areas lost and gained in doing so.

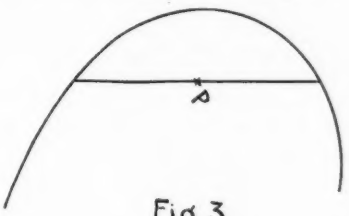


Fig. 3

The theorem is essentially sound, but a precision of the statement is obviously necessary (for example, for a closed curve, also a maximum is cut off by the chord). The theorem is fascinating on account of the lack of restrictions on the curve. If I may venture an opinion, it has possibilities.

As an immediate application, one obtains a proof of the theorem that for an ellipse and a given point P inside the ellipse, the chord through P cutting off the minimum (and the maximum) area is parallel to the diameter conjugate to the diameter through P.

[1] See J. R. Musselman, *On certain types of polygons*, this MONTHLY, March, 1933, for this and related theorems. (Added in proof).

(*b*). A couple of years ago this MONTHLY offered a problem; *Given an ellipse, with foci F_1, F_2. Draw any straight line from F_1 to P_1 on the ellipse, then draw P_1F_2 to P_2 on the curve, then P_2F_1 to P_3, P_3F_2 to P_4, etc. The segments will then approach the major axis as a limiting position.*

In contrast to the last theorem, this one is rather remarkable for the unnecessary restrictions placed on it. At least the following is true; *Given any oval, and any two interior points. If we carry out the construction as indicated the segment will always approach the segment joining the two given points as a limiting position.* This problem also seems to offer some interesting extensions.

A long time ago I discovered a theorem belonging to the large group connected with the Gauss-Lucas theorem on the roots of an algebraic equation and the roots of the derived equation.

If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are in the plane of complex numbers the points marking the roots of an algebraic equation, $f(z)=0$, then the roots of $f'' \cdot f - f' \cdot f' = 0$ all lie in the region of the complex plane bounded by the curve from all points of which the smallest convex polygon around $\alpha_1, \alpha_2, \cdots, \alpha_n$ appears under a right angle.

For a given system of n points, the construction of the curve is trivial:

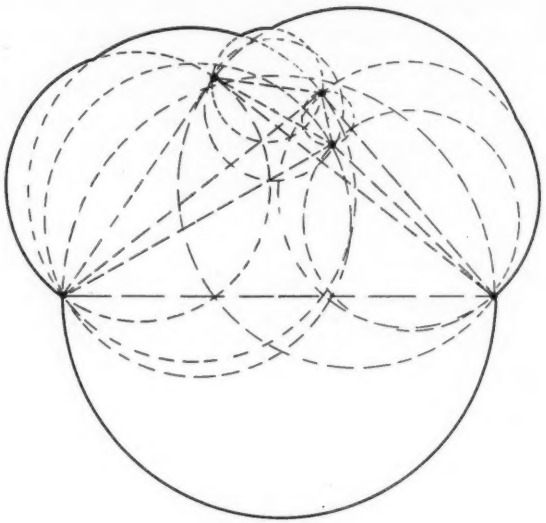


Fig. 4

As a corollary, we have:

If all roots of an algebraic equation lie on a segment of a straight line (for example, all roots real), then the roots of $f \cdot f'' - f' \cdot f' = 0$ lie inside the circle with the segment as diameter.

As illustration; For a Legendre polynomial $P_n(z)$ it is known that all the roots of $P_n(z)=0$ are real and between -1 and $+1$. Therefore, all roots of $P_n' \cdot P_n' - P_n \cdot P_n'' = 0$ lie in the unit circle. Also, all roots of $P_n^2(z^2+n+1) - 2zP_nP_{n-1}(n+1)+nP_{n-1}=0$ lie in the unit circle.

Papers published since then by various authors contain theorems in the statement of which occur the loci from the points of which some given region appears under a constant assigned angle.

This type of problem seems to me worthy of some attention, because, on the one hand, it has not, as far as I am aware, been systematically studied, while, on the other hand, some very special cases are well known as independent theorems. For conic sections, in particular, the director circles of the ellipse and hyperbola are the curves, from all points of which these curves appear under right angles; for the parabola, it is the directrix. When the angle is different from $\pi/2$ the locus, for the parabola, is a certain hyperbola (really one branch of the hyperbola); for the ellipse and hyperbola, certain curves of the fourth order.

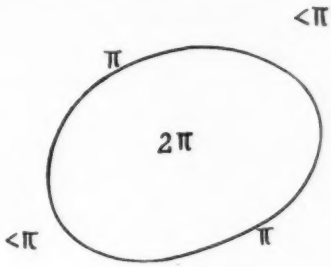


Fig. 5a

For interior points, it is natural to define the angle under which the region appears, as 2π. For the boundary points on an oval, the angle is π. For points outside an oval, the angle is smaller than π. For regions which are not oval, the situation is slightly more complicated:

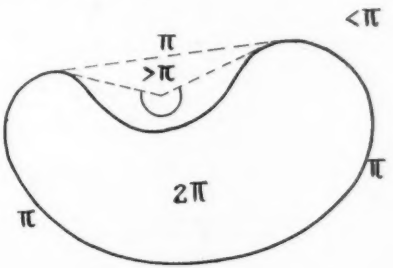


Fig. 5b

For the broken line formed by two rays making an angle α the angle under which this broken line appears is an interesting, although perfectly simple, function of the position of the point. Along any circle with P as center, we have an arc (of angle α) along which our angle is constant ($=2\pi-\alpha$), and independ-

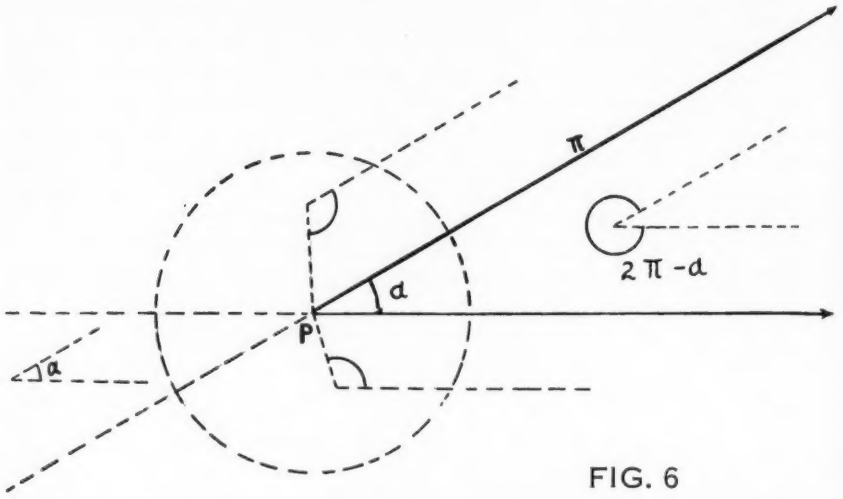


FIG. 6

ent of the radius; then an arc (of angle $\pi-\alpha$) along which it changes uniformly from π to α; then an arc (of angle α) for which the angle is constant ($=\alpha$); finally an arc (of angle $\pi-\alpha$) along which the angle changes from α to π.

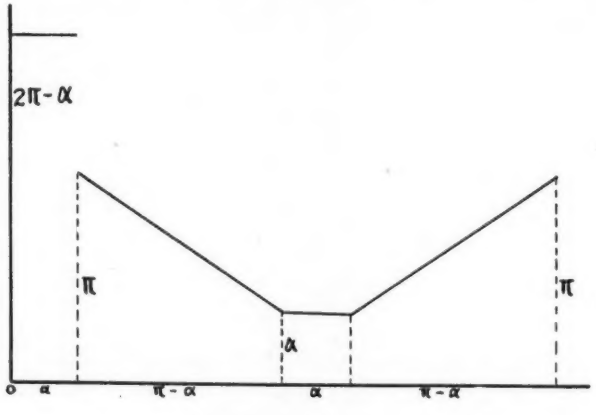


Fig 7

III. *Functional Dependence.*

The classical criterion for functional dependence of n functions of n variables, $f_i(x_1, \cdots, x_n)$, $i=1, 2, \cdots, n$, is the identical vanishing of the Jacobian,

$$\frac{\partial(f_1, \cdots, f_n)}{\partial(x_1, \cdots, x_n)} = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}$$

Hence, $2x+y^2$, $x-y$ are functionally independent, and $(x-y)^3$, $(x-y)^2$ are functionally dependent. The definition for two functions of two variables, $u=f_1(x, y)$, $v=f_2(x, y)$ is: u and v are functionally dependent if v is a function of u alone, $v=\phi(u)$.

The drawback of the Jacobi criterion is, that so very many assumptions on the f_i have to be satisfied before it is applicable (existence of the n^2 partial derivatives, etc.)

As a matter of fact, it is well known that we may speak of functional dependence and independence even though the derivatives do not exist; even though the f_i be not continuous. One set of criteria, established within recent years, seems to have as its root such simple considerations as the following:

Consider the independent functions $u=x+y$, $v=x-y$. Any two-dimensional region in an xy-plane is by our functions plotted onto a two-dimensional region in the uv-plane.

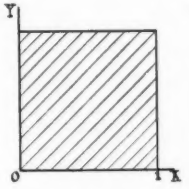


Fig. 8a

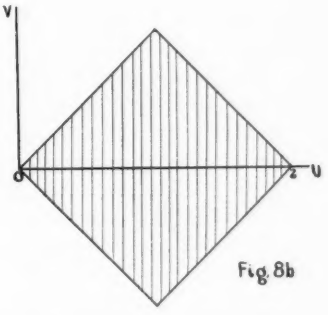


Fig. 8b

For the two dependent functions, $u=(x-y)^2$, $v=x-y$, $v^2=u$, a two-dimensional region in the xy-plane is transformed into a one-dimensional region, namely, a part of the parabola $v^2=u$ (See Fig. 9).

The final criteria are a good deal more complicated than this; but this seems the underlying idea. The method permits one to eliminate the restrictions imposed by the Jacobi criterion.

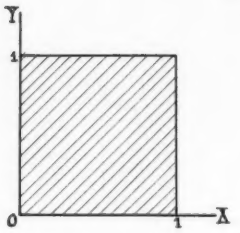


Fig. 9a

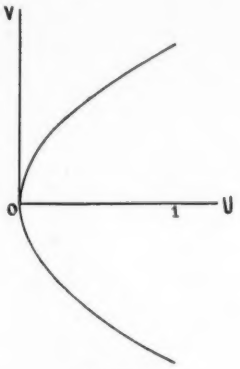


Fig. 9b

I should like to present to you another geometric interpretation of functional dependence, which, simple as it is, I have not met in the literature. I admit that I have not been able satisfactorily to extend it to the case $n>2$.

We recall, then, $u(x, y)$, $v(x, y)$ are functionally dependent if $v=\phi(u)$. Assume for u and for v the sets of niveau curves drawn in the xy-plane. Then if v is to be a function of u, not explicitly containing x or y, this will mean that, for any curve in the xy-plane along which u is constant, $u(x, y)=c_1$, $v(x, y)$ must also be constant, $v(x, y)=c_2$. But this means exactly that the two functions $u(x, y)$, $v(x, y)$ have the same system of niveau curves.

Theorem: $u(x, y)$, $v(x, y)$ *are functionally dependent when and only when they have the same system of niveau curves.*

It is not necessary that the niveau curves be in one-to-one correspondence in the two sets.

If the four partial derivatives exist, the vanishing of the Jacobian is easily interpreted in the theorem. However, u and v may have the same system of niveau curves even when some or all the derivatives fail to exist. For example, assume first a function $\phi(x, y)=x^2+y^2$; the niveau curves will be the set of concentric circles about the origin. The cross section of ϕ with the $x\phi$-plane is a parabola. By moving cross-sections up and down along the ϕ axis, and rearranging their order, we may change our surface into one which is still a surface of revolution, but the cross-section may be made a monotonically increasing, nowhere differentiable curve; indicated schematically in Fig. 10b. Call this function $u(x, y)$. Then $\partial u/\partial x$, $\partial u/\partial y$ will not exist. Now build up from $\phi(x, y)$ by the same process another surface $v(x, y)$, which enjoys the same properties. u and v have the same system of niveau curves, and it will be entirely proper and meaningful, to say that $u(x, y)$, $v(x, y)$ are functionally dependent, although the Jacobian does not exist.

Indeed, applying the Dirichlet definition of a function, over some assigned range of the arguments x and y, the criterion will still apply to functions de-

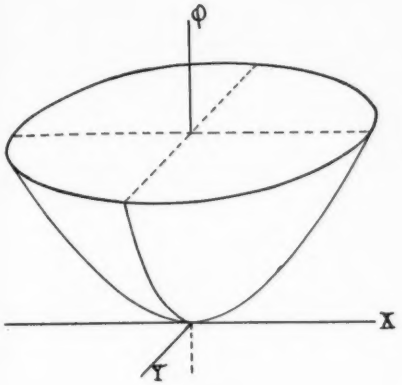


Fig. 10a

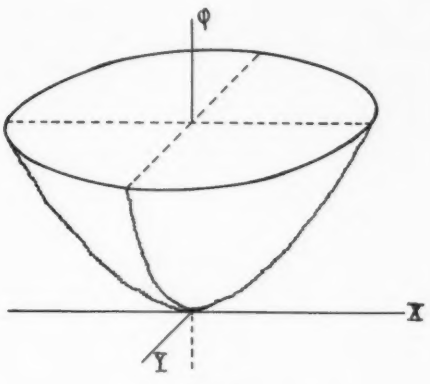


Fig. 10b

fined only for a discreet set of points in the xy-plane, as in the case of many number-theoretic functions, or even to functions which are defined for only a finite number of argument pairs.

IV. *The Geometric Representation of Imaginaries and the Isotropic Lines.*

The circular points are at indeterminate distance from finite points, and are imaginary. Nobody is scared by either one of these attributes, but the combination is a trifle overpowering to the unsophisticated mind. In homogeneous coordinates in the plane, the two circular points may be written $(1, i, 0)$, $(1, -i, 0)$. One of the outstanding properties of these two points is, that every circle in the plane passes through them. This follows as the result of very simple algebraic considerations. It is really very easy to make this startling statement plausible to an undergraduate student, but my geometrical friends frown on the method —which has been used, in other connections, over and over again—as being artificial. Our ordinary representation in a plane of $y=f(x)$ obviously strictly excludes all but reals; for the complete representation of complex values of both variables by means of individual points, we need either a four-dimensional Euclidean space, which has obvious disadvantages, or two two-dimensional planes, as in conformal mapping. The latter method, elegant and powerful as it is, sacrifices completely the shape of the real curve: one does not obtain it from conformal mapping by specialization, since one has only, as far as reals are concerned, a correspondence between the two axes of reals, or segments of them. One can, however, make a very valuable compromise by allowing one variable, say x, to assume complex values, while restricting y to real values. Then, using three-dimensional space, with $x=x_1+ix_2$, one will determine all complex values x for which y is real, and will thus obtain a space curve which gives in the x_1y-plane the ordinary real curve, with a lot of valuable additional information besides.

For a circle $(x-a)^2+(y-b)^2=r^2$, we have in the x_1y-plane the ordinary real circle; at the highest and lowest points are attached the branches of an equilateral hyperbola, which lies in a plane perpendicular to the plane of the circle.

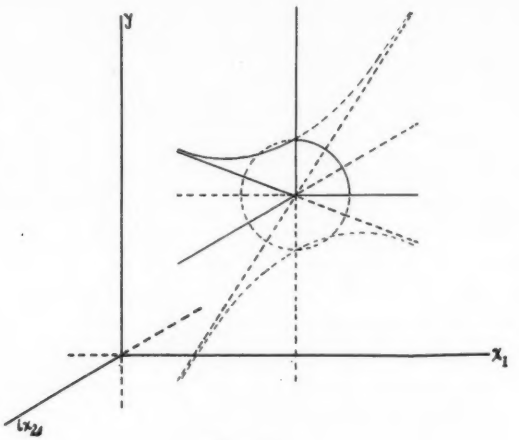


Fig. 11

These hyperbolic handles should always be thought of as forming part of the circle. For an ellipse, we obtain the same kind of figure, with hyperbola branches attached at the highest and lowest points, but the hyperbola is now no longer equilateral. We clearly have something intermediate between our ordinary real graph and the unattainable ideal of a simple representation of all complex x and y satisfying the equation.

It is then obvious, from our customary manner of considering infinite elements, that all circles pass through the circular points. I only mention in passing that any real polynomial curve yields very gracefully to similar treatment, with remarkable results. (Phillips and Beebe, *Graphic Algebra*, make much use of it in the study of some particularly simple curves, but with no attempt at a systematic treatment.)

For the cosine (or sine) curve, as limiting case of a polynomial curve, one has a particularly pretty result. The handles attached to the curve, are plane catenary curves (See Fig. 12).

This method of construction is equivalent to either of the following:

(*a*) Over the $x=x_1+ix_2$ plane erect the real surface of absolute values, $|y|=|f(x)|$, and also the surface $-|f(x)|$, which is symmetric to the first with respect to the x-plane. Then construct over the x-plane the surface u of real parts of $y=u+iv=f(x)$. Our space curve is the curve of intersection of these two surfaces, as is at once seen.

(*b*). For conic sections, the construction leads, except for a rotation through

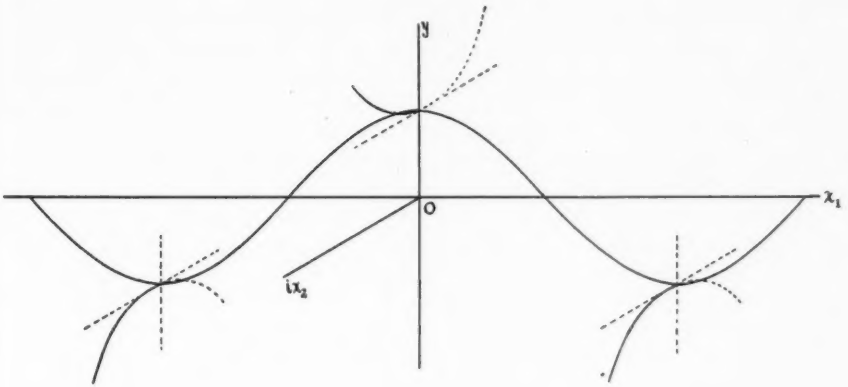


Fig. 12

90° into a different plane, to the figure of Poinsot and his "cordes idéaux." Poinsot's construction for $x^2/a^2+y^2/b^2=1$: For a real chord M_1M_2, the pole P_2 is constructed by means of the tangents; then P_1P_2 harmonic to A_1A_2, and $\overline{P_1M_1}^2 = p \cdot P_1A_1 \cdot P_1A_2$, p a constant which does not interest us here. For a parallel chord not intersecting the ellipse in real points, intersecting A_1A_2 at P_2, we carry out the corresponding constructions, finding now P_1 from P_1P_2 harmonic to A_1A_2, and determining N_1 (and N_2) from $\overline{P_2N_1}^2 (=\overline{P_2N_2}^2) = p \cdot P_2A_1 \cdot P_2A_2$,

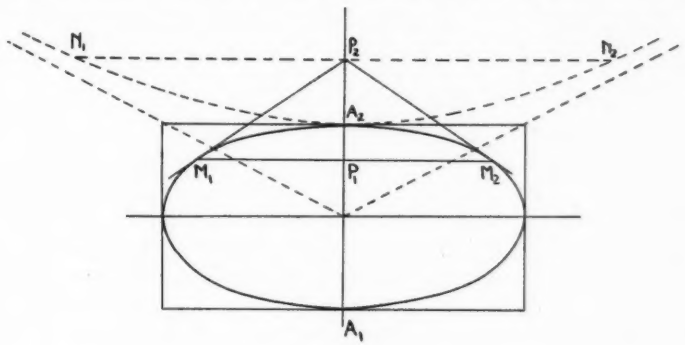


Fig. 13

with the same p as above. The locus of N_1 (N_2) is an hyperbola with the same conjugate diameter as the ellipse. Our figure for the ellipse is exactly the same as this Poinsot figure, but with the hyperbola rotated around the axis A_1A_2 by an angle $\pi/2$.

I recall my own difficulties in connection with complex points on algebraic curves when Professor Klein of Göttingen would draw on the board a curve, mark a point P on it, and say: "Dies ist ein komplexer Punkt auf der Kurve." I think I should have understood better if it had been pointed out that a curve, say for example $y=ax+b$, is, for complex x and y, a two-dimensional continuum, in four-dimensional space. We have $y_1+iy_2=a(x_1+ix_2)+b$, $y_1=ax_1+b$, $y_2=ax_2$; each of these is a three-dimensional complex in the four-dimensional $x_1x_2y_1y_2$-space, their "intersection" a two-dimensional domain in the space.

Now consider one of the isotropic lines through the origin, i.e., a line from the origin to one of the circular points, $(1, i, 0)$. In non-homogeneous form, this is the line $y=ix$, while the other isotropic line through the origin is $y=-ix$. Similarly, for a point different from the origin, $y=ix+c$, $y=-ix+c$.

I wonder whether students are still allowed to think that "an isotropic line is perpendicular to itself?" Assume, without real loss of generality, the line to be $y=ix$. Then

Proof: $i=-1/i$, q.e.d.

This proof may claim to be the shortest, neatest, wrongest, proof in existence.

There are, of course, many books in which the situation is properly explained in terms of "indeterminate" quantities, etc. But an analyst wonders why it is never (at least, as far as I am aware) mentioned that the equation $i=\tan\phi$, which we obtain by identifying the coefficient of x with the slope of the line, has no meaning because i is a Picard exceptional value of the analytic function $\tan z$, that is, a value which is not assumed by $\tan z$ for any value of z, real or imaginary, finite or infinite. ($e^z=0$, known impossible, because 0 is an exceptional value for the exponential function; then $\tan z=(e^{2iz}-1)/(ie^{2iz}+i)=i$, $i\cdot e^{2iz}+i=-i\cdot e^{2iz}+i$, $e^{2iz}=0$, $e^{iz}=0$, $e^{z_1}=0$, for $z_1=iz$, impossible.)

The misunderstanding arises not from speaking of the line $y=ix+c$—although it is a queer line—, but from identifying the coefficient i with the slope $\tan\phi$ and applying to it the regular rules for slopes.

It is true that if one takes $y=\gamma x+c$, $\gamma=i+\epsilon$, $\epsilon\to 0$, ϵ real or complex, then

$$-\frac{1}{i+\epsilon}=-\frac{i-\epsilon}{i^2-\epsilon^2}=\frac{i-\epsilon}{1+\epsilon^2}\to i,$$

and that therefore, if the coefficient γ is close to i, a line perpendicular to $y=\gamma x+c$ will have its coefficient close to i, also. But they will not be equal! For real slopes, the difference $m-(-1/m)=m+1/m$ also varies with m; for real positive m, its minimum is 2, and is attained for $m=1$. For complex m, $m-(-1/m)$ may have any value, including 0 for $m=\pm i$, but for these values there is no angle of inclination!

It is, indeed, just as easy to say that an isotropic line makes any assigned angle with itself, and just as sensible; and one does sometimes read this statement.

We simply have an illustration of the very familiar state of affairs that a

theorem which holds for a variable, may break down when the variable reaches a limiting value.

Another well known property of the isotropic lines is that on such a line the distance between any two points is zero. I only state that this, bizarre as it first sounds, involves no logical conflict. The queerness arises solely from the fact that, for real lines, the function $d^2=(x_2-x_1)^2+(y_2-y_1)^2$ vanishes when *and only when* $x_2=x_1$, $y_2=y_1$, so that for real lines a distance zero is equivalent to saying that two points coincide, while along an imaginary line $(x_2-x_1)^2+(y_2-y_1)^2$ may vanish without $x_2=x_1$, $y_2=y_1$:

For real lines $d=0$ is necessary and sufficient for coincidence of two points; for imaginary lines it is necessary, but not sufficient.

If it were desired to retain, for complex lines, the fact that $d=0$ be a necessary and sufficient condition for coincidence, the distance could be defined as the non-negative value D given by

$$D^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2,$$

which would, of course, for real lines, agree with our ordinary definition.

One remark concerning the geometric representation: To represent a line containing two general complex points $(\alpha_1+i\beta_1,\ \gamma_1+i\delta_1)$, $(\alpha_2+i\beta_2,\ \gamma_2+i\delta_2)$ requires, as we know, a four-dimensional space or two two-dimensional spaces, unless we fall back on some intrinsically complicated method. But a line containing two points $(\alpha, i\beta)$, $(\gamma, i\delta)$ may be naturally represented as a line in an ordinary plane of complex numbers. Its equation is $y(\gamma-\alpha)+x\cdot i(\beta-\delta)+i(\alpha\delta-\beta\gamma)=0$.[1] Using our standard definition, we have $d^2=(\alpha-\gamma)^2-(\beta-\delta)^2$; using $D^2=|x_2-x_1|^2+|y_2-y_1|^2$, we have $D^2=(\alpha-\gamma)^2+(\beta-\delta)^2$.

V. *The Shape of Polynomial Curves.*

Geometers and algebraists will agree that the straight line, given by a linear equation is the simplest of all curves. When one asks for the type next in simplicity, I should rather expect the geometer to select the conic sections, while the algebraist may be more inclined to select the polynomial curves, $y=\sum_{i=1}^{n}a_ix^i$, on account of the algebraic simplicity of these functions. Thus the parabola, which belongs to both classes, would indicate the parting of the ways. However this may be, it seems remarkable that some really fundamental questions concerning polynomial curves are apparently unanswered.

Altogether, we do not seem to know too much about the geometry of such polynomial curves. Such knowledge as we do possess, is obtained by algebraic methods as a consequence of the fact that our polynomial curves are algebraically defined. I do not recall ever having heard of a characteristic geometric property of polynomial curves, except, possibly, the fact that if one forms successive derived curves, (which can be constructed geometrically point by point

[1] Of course there is also an infinite number of "points" $(\xi, i\eta)$ with complex ξ, η satisfying the equation; just as in ordinary analytic geometry the equation $Ax+By+C=0$ is satisfied by an infinite number of complex x, y.

from the given curve) one will ultimately obtain a straight line—just as the exponential curve $y=e^x$ may be defined by the property that it coincides with its derived curve, or that the curve of the nth differences is a horizontal line.

That even such knowledge as we do possess, is not so widely spread as it should be, is evident from the frequency with which one finds, for example, a polynomial cubic curve represented by a curve of the type shown in Fig. 14a instead of that in Fig. 14b.

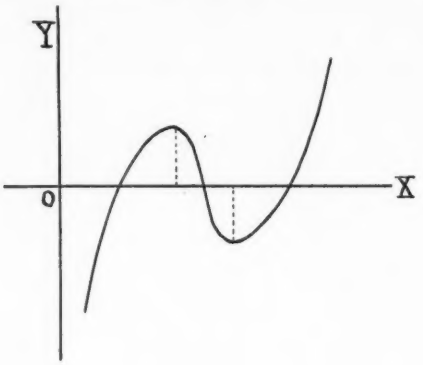


Fig. 14a

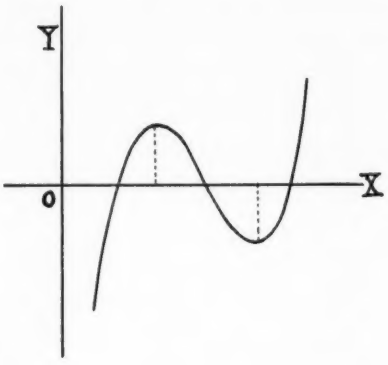


Fig. 14b

Of course, any straight line cuts such a polynomial curve of the nth degree in exactly n points, if proper allowance is made for coinciding, for infinite and for complex points of intersection. Then, one well known theorem states that, for $n>2$, for all straight lines the arithmetic mean of the abscissae of the points of intersection of the line with the given curve is constant. This is an obvious interpretation of

$$y = f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,\ n > 2$$

$$y = ax + b,$$

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-2}x^2 + (a_{n-1} - a)x + (a_n - b) = 0,$$

$$\sum x_i = -a_1/a_0,$$

regardless of the values of a and b. Similarly: Given $y=f(x)$, $y=\phi(x)=b_0x^m+\cdots+b_m$, $m\leqq n-2$. For the points of intersection of any $\phi(x)$ with the $f(x)$, $\sum x_i$ is constant.

And: Given $y=f(x)$ of degree n, $\phi(x)=b_0x^m+\cdots+b_{m-1}x$, $m\leqq n-1$. For the points of intersection of any $\phi(x)$ with the given $f(x)$, the product πx_i is constant.

Is it not rather surprizing that existence theorems of the following type have apparently completely escaped attention?

Assume for concreteness $n=5$, $a_0>0$,

$$y = a_0x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5.$$

Assume first that $f(x)$ has its maximum number of extremes, four, for $x_1>x_2>x_3>x_4$. Certain inequalities are trivial: $f(x_1)<f(x_2)$, $f(x_2)>f(x_3)$, $f(x_3)<f(x_4)$.

Are there any further restrictions on the order of magnitude in which $f(x_1)$, $f(x_2)$, $f(x_3)$, $f(x_4)$ follow each other, or are all cases, consistent with the trivial inequalities, actually realized for some $y=a_0x^5+\cdots+a_5$?

We will for simplicity assume that no $f(x_i)=f(x_j)$. Consistent with the trivial inequalities are exactly the five types shown in Fig. 15.

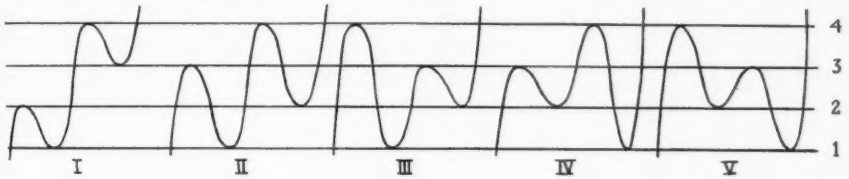


Fig. 15

For a given n, the establishment of the number of types is given by a neat little problem in the theory of numbers.

It is understood that the figures are schematic, in that nothing is stipulated concerning the actual distance of the parallel lines from one another.

The types are easily indicated by the symbols (3412), etc., as in the figure.

This problem interested me some years ago, and I succeeded in showing that *all conceivable types really exist for any degree.*[1]

For example, a real curve of order 6, of type (45132), is represented by

$$y = (x + ia\epsilon)(x - ia\epsilon)(x - a)^2(x - a - a\epsilon^2 + ia\epsilon^3)(x - a - a\epsilon^2 - ia\epsilon^3),$$

a arbitrary, >0; $\epsilon>0$, sufficiently small.

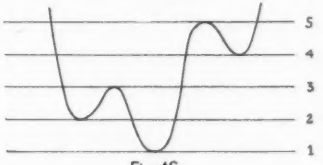


Fig. 16

I repeat that I cannot prescribe the distances between the successive lines 1, 2, 3, 4, 5. The kind of curve the method yields will have each one of the

[1] See Kempner, *On the shape of polynomial curves*, Tôhoku Mathematical Journal, vol. 37 (1933), pp. 347–362.

following four distances large as compared with the succeeding one: 15, 14, 13, 12.

The problem becomes more complicated when the $f(x)$ does not possess the maximum number $n-1$ of extremes. This may happen by having two consecutive maxima and minima coalesce and form a point of inflexion, with horizontal tangent. Further varying the constants of the polynomial, the tangent of inflexion may be made to assume a slope different from zero, while further variation of the coefficients may cause the point of inflexion to vanish entirely.

For $n=5$, $a_0>0$, this leads to eight additional types shown in Fig. 17.

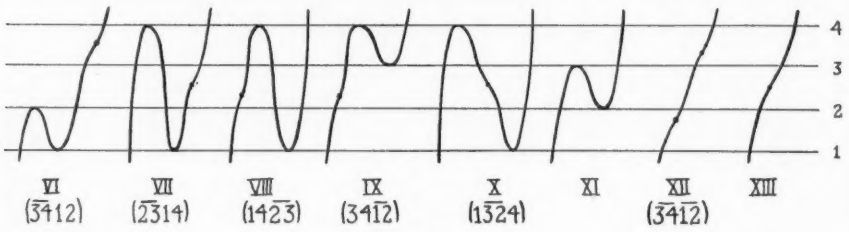


Fig. 17

It is fortunately very easy to distinguish the new points of inflexion from the points of inflexion which necessarily occur between two successive extremes. If I may for the moment call the latter type "natural" points of inflexion, and the former, "accessory" points of inflexion, it is at once seen that, at a natural point of inflexion, $|f'(x)|$ is a *maximum*, while at an accessory point, $|f'(x)|$ is a *minimum*.

It is not natural to prescribe any restrictions on the natural points of inflexion: they are sufficiently determined by the fact that there is exactly one of them between two consecutive points A, where the set of A points consists of the extremes and the accessory points of inflexion.

The notation employed for curves of types VI, VII, VIII, IX, X, XII [$(\bar{3}412)$ etc.] is immediately explained by reference to I–V.

Again, all possible types can be realized by polynomial curves of degree n.

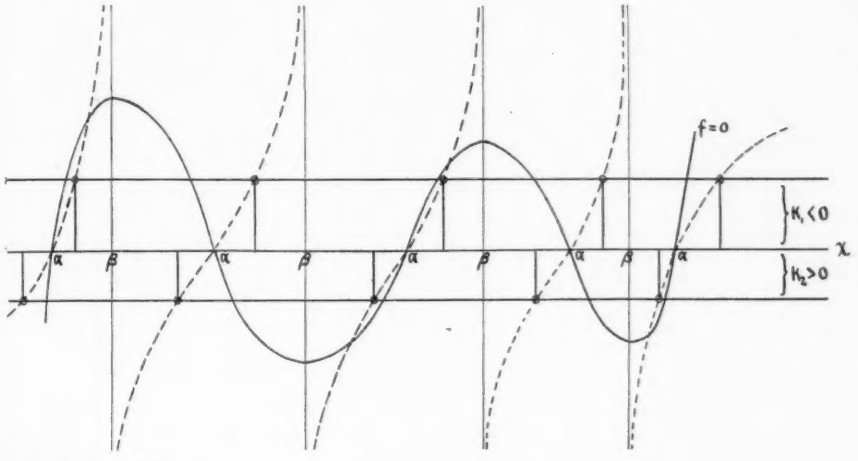
The method of attack offers no additional difficulty when applied to curves with fewer than the largest possible number of extremes (type XI, for $n=5$).

VI. The last question which I shall mention deals with a type of problem where a large number of algebraic theorems, which have at various times been proved independently by algebraic methods, all follow immediately from such simple geometrical facts as that, if for $a \leqq x \leqq b$, $y=f(x)$ is on opposite sides of the x-axis for $x=a$ and $x=b$, the continuous curve $f(x)$ must cross the axis at least once in the interval.

Consider, for example, the very simple and very well known theorem: If $y=f(x)$ is a polynomial with all zeros real and distinct, and k a real constant $\neq 0$, then $f+kf'=0$, i.e. $f/f'=-k$, has all roots real and distinct, and they sepa-

rate the roots of $f=0$ and are separated by them. (We may add: They also separate the roots of $f'=0$.)

Figure 18 explains itself.



Dotted curve: f/f'

α: roots of f = 0

β: roots of f' = 0

Fig. 18

The point I wish to make is this: I am quite aware of the very justifiable disinclination of mathematicians to rely on geometric intuition for proofs. But caution may be carried to an extreme. In the present case the theorem is really very much more general than is indicated, since it holds for all functions whose graphs are of the type: $f(x)$ continuous, and with continuous first and second derivatives; maxima and minima alternating in sign, and between two extremes only one point of inflexion. The polynomial curves to which the theorem is applied form but a very small subset of these curves.

I have already mentioned that frequently a considerable number of such algebraic theorems are nothing but special cases of an appropriately selected geometrical theorem.

To this one may add: The geometrical theorem may frequently yield at a glance additional information not immediately suspected in the algebraic proof. In our theorem: For $k>0$ the separation of the roots of $f+kf'=0$ by the roots of $f=0$, or $f'=0$, is of an easily stated different type than for $k<0$.

These topics represent the type of work I have in mind when I refer to geometry as an avocation.

MATHEMATICAL INTERPRETATIONS OF GEOMETRICAL AND PHYSICAL PHENOMENA[1]

By G. A. BLISS, University of Chicago

Mathematical theories have been of great service in many experimental sciences in correlating the results of observations and in predicting new data afterwards verified by observation. This has happened particularly in geometry, physics, and astronomy. But the relationship between a mathematical theory and the data which it is designed to relate is often misunderstood. When such a theory has been successful as a correlating agent the conviction is likely to become established that the theory has a unique relationship to nature as interpreted for us by the observations. Furthermore, it is sometimes inferred that nature behaves in precisely the way which the mathematics indicates. As a matter of fact, nature never does behave in this way, and there are always more mathematical theories than one whose results depart from a given set of data by less than the errors of observation.

The purpose of these lectures is to explain in more detail the point of view which is indicated briefly in the preceding paragraph. A description will be given of the structures of pure and applied mathematical sciences. For this purpose the most accessible mathematical domain is geometry, and the first few paragraphs below are devoted to a discussion of various geometrical theories. But examples from astronomy and physics will also be briefly described in later sections.

1. *Geometrical measurements.* If our geometrical experiences were only those of very ancient peoples our collection of observed geometric data would be a very limited one. Nowadays, however, we have many sources to draw from. We accumulate geometric data by measurements on the drawing table, by local and geodetic surveys, by observing the motions of the planets, or even by the relatively very rough measurements of stellar distances which are now in use. In every case the measurements have a percentage of inaccuracy which can be experimentally determined, and in every case the domain of observation is of finite extent.

On the drawing table, for example, we designate as points the dots which we make with our pencil, and straight lines are the marks which we make by drawing the pencil along the edge of a ruler. It is not true experimentally, as we should perhaps like to have it, that every pair of dots is joined by one and only one straight line. If we were to sharpen our pencil sufficiently, and perhaps use a magnifying glass, we should see that a pair of dots can be joined by very many delicately drawn straight lines. It is likewise not true experimentally that the sum of the angles of a triangle is always 180 degrees. If we measure our angles honestly we find that the sum is usually not exactly 180 degrees, but that the

[1] Two lectures delivered in the General Course of the Division of Physical Sciences of the University of Chicago.

maximum variation of the sum from that amount is a small number which depends upon the instruments used and the physical characteristics of the observer. Other observations which may be cited to illustrate the inaccuracy which attends all measurement are the measurements of stellar distances. The methods used are too technical to be described here. The points to be emphasized with regard to them are that even these enormous and almost incomprehensible distances are still finite, and that the percentage of inaccuracy in their determination is very great. That they can be estimated at all is a great tribute to the ingenuity of astronomers.

2. *Fundamental postulates of geometry.*[1] In view of the inaccuracy of geometrical observations it is clear that we can never hope to find a precise mathematical theory to correlate them which will be an exact fit. Until recently it has been customary to idealize points and lines. A point was something which had no dimensions, and a line had only one dimension, namely, length. We may still find such definitions in our text books. Unfortunately our observational experience does not provide us with any such geometrical entities. The dots and straight line marks of the drawing table certainly have dimensions, and it is, after all, their properties which our mathematical theory, whatever it may be, should correlate approximately.

The procedure nowadays is quite different. We do not attempt to idealize points and lines. We rather try to construct an ideal geometrical theory by idealizing the fundamental properties which we ascribe to points and lines. The statements of these fundamental properties are called postulates, and from them, by the processes of mathematical logic, are deduced the theorems which constitute the exact science of geometry. Thus in our ideal geometry one of the postulates is that two points can be joined by one and but one straight line. The dots and straight line marks on the drawing table do not satisfy this postulate, but the dots can be made so small that the eye alone cannot distinguish between the lines which join them. A theorem in our exact geometrical theory, based upon the postulate, should therefore correspond to a property of the dots and marks with a degree of approximation similar to that with which the postulate itself is applicable. Described somewhat roughly the theorem should seem true to the unaided eye for dots and marks when the dots are made sufficiently small.

The pure mathematician is of course primarily interested in exact abstract geometrical science, but he is guided in his selection of postulates by his observational experience. It is impossible here to give a complete list of the postulates for geometry. As formulated by the German mathematician, Hilbert, they are twenty in number, and are of five types which are called the postulates of connection, order, congruence, parallelism, and continuity. Examples of each of these will be given in the following paragraphs.

The postulate mentioned above, stating that through two points there passes one and but one line, is a postulate of connection. There are six others of

[1] Hilbert, *Foundations of Geometry*, Open Court Publishing Co., 1902; Veblen and Young, *Projective Geometry*, vol. I, Introduction and Chapter I.

this type. If three points lie on the same line then one and only one of them lies between the other two. This is a postulate concerning order. It seems self-evident unless we recognize, for example, that of three lines through a point each one may be regarded as lying between the other two. By this example we see that there are other types of order besides that described in the postulate concerning the order of the points on a line, and that therefore some agreement about the order of such points is necessary if our geometry is to have a sound basis. There are in all five postulates concerning order. If two triangles have two sides and the included angle of the one equal to two sides and the included angle of the other then the triangles are congruent. This postulate is the basis of that part of geometry which has to do with congruent triangles. It is a sample of the postulates of congruence which are six in number.

Now we come to the famous assumption of Euclid concerning parallel lines. If a line l and a point P are given, then there is one and but one line through P which has no point in common with l. This is not by any means a self-evident property of the straight lines of our experience, though geometers for many centuries thought that it was so or that it could be proved to be a consequence of other simple postulates. No one was able to make such a proof, and when another geometrical theory was found with this postulate changed, but all the others just the same, it became evident that no proof of the dependence of Euclid's postulate could ever be made. The possibility of changing the postulate to another is a consequence of the facts noted above that all of our geometrical experiences are in a finite portion of space, and that our geometrical measurements agree only to within a certain percentage of error. This will be clearer after we have considered in the next section an example of a geometry in which Euclid's postulate is not true, but which nevertheless correlates the facts of our geometrical experience with as high a degree of accuracy as does Euclid's geometry itself.

The final postulate which we shall consider, the so-called postulate of continuity, is usually ascribed to the famous old Greek scientist, Archimedes. If A, A_1, and B are three points in that order on a line then there is always a sequence of points A, A_1, A_2, $\cdots$, A_n such that the segments between adjacent points are all congruent, and furthermore such that B lies between A and A_n. This is equivalent to saying that by starting at A and laying off successively equal segments we can always ultimately pass the point B.

It will be noted that all of the postulates which have been stated above are in accord with our geometrical instincts. They and the others, which for want of space could not be mentioned, are statements of simple properties of points and lines which we intuitively willingly adopt as proper bases for a geometrical theory, and which agree to a suitable degree of approximation with geometrical data on the drawing table or in any other finite region where such data are obtainable. It is clear that none of us can follow two lines out to more than a very limited distance to see if they ever meet, and none of us can check the postulate of Archimedes for points B which are very far away. We select these postulates

because we like them and think that they will help to give us a useful theory.

3. *A simple non-Euclidean geometry.* Examples of sets of geometrical data which can be correlated by two different geometrical theories are very common. The surveyor, for example, uses the formulas of plane geometry when he measures the area of a city lot or of a farm. Theoretically he should of course use spherical geometry, since his measurements are made on the spherical surface of the earth. It would be foolish of him to do so, however, since the formulas of plane geometry are much easier to work with than those of spherical geometry, and since for a small area the surface of the earth is so nearly a plane that no ordinary surveying instrument could ever distinguish between the two. Evidently in this case either of the two theories would give results in agreement with observed data to a high degree of precision. The choice between the two for practical work is made for reasons of simplicity and convenience.

Just as the surveyor uses plane instead of spherical formulas for measurements on a sphere, so one may interpret plane geometrical phenomena by a spherical theory. To see this let us consider a very large sphere resting on a plane table-top. We may associate with each point P on the plane the point on the sphere which is the intersection of the straight line joining P to the center of the sphere. The points of a segment of a straight line in the plane then correspond to those of a segment of a great circle on the sphere. If we define the length of the plane segment as the length of the corresponding great circle segment on the sphere, and make a similar definition for the measurement of angles, a new geometrical theory for the plane can be constructed.

It should be noted first of all that when the sphere is sufficiently large the new measures for segments and angles on the plane agree with those of ordinary Euclidean geometry so closely that no measuring instrument, however delicate, could ever distinguish between them. Every straight line has a finite length, in the new geometry, equal to the length of half of a great circle on the sphere. The sum of the angles of a triangle on the plane is greater than 180 degrees, since the sum of the angles of a triangle on the sphere has this property. Finally we see that through a point P outside of a line l there passes no line parallel to l. Every pair of lines in the plane have a common point at a finite distance, since every pair of great circles on the sphere has a point of intersection. Thus Euclid's postulate is not true in the new geometry. These discrepancies with Euclid's theory need not disturb us at all, however, as far as our interpretations of geometrical data on the table-top are concerned. In this limited domain the two theories agree so closely that either one offers an acceptable explanation. We should be foolish to select the spherical theory for practical purposes, however, since the formulas of spherical geometry are much more complicated than those of Euclidean geometry in the plane.

In concluding our consideration of this example it should be stated that there are many other mathematical theories for the interpretation of geometrical phenomena in the plane. The theory which has been suggested above was discovered by Riemann and is sometimes called Riemannian non-Euclidean geom-

etry. There are also geometries in which through each point P not on a line l there pass an infinity of lines not meeting l. These lines through P form a sheaf so thin that it is not distinguishable experimentally from a single line.

4. *The structure of a mathematical science.* A pure mathematical science consists of postulates, definitions, and theorems. Thus for geometry we agree first of all that the names of elements to be studied shall be points and lines, and we postulate a set of simple properties for these elements from which all other properties are to be deduced. We define in terms of points and lines what we mean by an angle, a triangle, a polygon, or a circle, and starting from the postulates we prove by processes of mathematical logic the theorems concerning the things so defined which constitute the important results of geometrical science.

The science of arithmetic has a similar structure. The postulates for positive integers are four in number.[1] For two such integers we can define what we mean by the sum, product, difference, and quotient, and we can prove for these the well-known laws of computation, such as the theorem that the product of a number of integers has the same value irrespective of the order in which the integers are taken. Furthermore, starting from the positive integers and their properties we can define and analyze the properties of the complete system of positive and negative real numbers. Thus we see again that arithmetic is a science consisting of postulates, definitions, and theorems.

An applied mathematical science has a somewhat more complicated structure. The physicist, astronomer, or geometer frequently finds himself in possession of a set of data, obtained by observation, which he would like to correlate by means of a mathematical theory of the kind described in the two preceding paragraphs. For this purpose he must first of all select a set of postulates, as simple as possible, which are in agreement with the observed data as nearly as the observations are in agreement with themselves. When the logical consequences of the postulates have been worked out, he must devise new experiments if necessary to check again with observation his theoretical results. Thus the structure of an applied mathematical science can be suggested by the following table:

Observed data

Postulates	Postulates	. . .
Definitions	Definitions	. . .
Theorems	Theorems	. . .

Check with observed data

The form of the table indicates that for the correlation of observed data it is to be expected that more than one mathematical theory may be effective, as we have seen above in the case of geometry. The selection of one among the theories possible is based on convenience, or on accuracy of fit with the data, or both.

[1] See, for example, Landau, *Grundlagen der Analysis*, Chapter I.

The danger is always when a theory has been found to be convenient and effective over a long period of time, that people begin to think that nature herself behaves precisely in the way which is indicated by the theory. This is never the case, and the belief that it is so may close our minds to other possible theories and be a serious impedence to progress in the development of our interpretations of the world around us.

Just as the same set of data may be related by more than one mathematical theory, so it is possible that the same theory may be effective in connection with more than one set of data. If we define a point to be a pair (x, y) of real numbers, and a straight line to be the totality of pairs (x, y) which satisfy an equation of the form $ax+by+c=0$, we may prove that these points and lines satisfy precisely the postulates of Euclidean plane geometry. Thus the science of geometry is applicable not only approximately to the dots and straight line marks of the drawing table, but also with exactitude to the number-pairs and linear equations of the Cartesian coordinate system. By the mechanism of such a coordinate system we thus bring to bear on geometry the powerful and familiar processes of arithmetic. The theory of geometry from this point of view is called analytic geometry.

In concluding this section it should be re-emphasized that the purposes of an applied mathematical science are two-fold, first to correlate and systematize data which may otherwise appear heterogeneous and unrelated in character, and second to predict by logical processes new results which might be difficult or impossible to discover by experimental methods alone. Many examples of the effectiveness of mathematical theories in both of these respects could be given if space permitted.

5. *Mathematical theories in astronomy.* The Ptolemaic and Copernican theories of the solar system are excellent examples of theories widely different in character but which with suitable modifications of the former describe with equal accuracy the motions of the planets. In the Ptolemaic theory the earth is regarded as the center of the solar system. The stars and the sun move on circles with the earth very near the centers, and in the modification suggested by Tycho Brahe the planets move on circles whose centers are at the sun and move with the sun. The curves thus described by the planets, as seen from the earth, are called epicycloids. This is the picture which we see when we stand out on the lawn and observe the heavens, and it has much to recommend it. Numerically it is not sufficiently accurate for astronomical purposes, but it could be made so by replacing the circles by ellipses, and superposing a suitable perturbation theory. In the familiar and more popular Copernican theory the sun is the central body of the solar system and the stars are fixed. The earth and the other planets move about the sun on ellipses each of which has the sun as a focus. The mathematical justification of the motions of the planets in either theory is based upon the famous law of gravitation of Newton.

There is really no advantage for either of these theories as compared with the other, as far as their adaptability to explain numerically the facts of the

solar system is concerned. The Copernican theory is, however, much the simpler geometrically and mathematically. For this reason it has been adopted and developed until astronomers can predict coming celestial events with most surprising accuracy, and it has resulted in the discovery of the two outmost planets, Neptune and Pluto.

But even the Copernican theory based upon Newtonian mechanics has failed to fit all of the observed data satisfactorily. The perihelion of a planet is the point nearest to the sun on its elliptical orbit. According to Newtonian mechanics the perihelion of the inmost planet, Mercury, should move about the sun each year a distance which differs from the observed motion by a perceptible amount. Many explanations for this discrepancy have been offered, but none of them was well received by astronomers until Einstein suggested a slight modification of Newtonian mechanics itself in accord with his so-called general theory of relativity. The new theory agrees with observed data, including the motion of the perihelion of Mercury, with discrepancies which are less than the errors of observation, but it is much less convenient for computation than the Newtonian theory. For most purposes we can still retain Newtonian mechanics since the two theories disagree by less than the errors of observation in all except a very few instances.

The theory of general relativity as applied to the solar system is already threatened, even before its usefulness has been completely established. Recently when Einstein and de Sitter were sojourning in the United States, the newspapers reported them as discussing still newer theories from which it might be concluded that the universe is finite. This conclusion should never be made. All that we could be justified in saying is that the data which we have concerning the distances and motions of the stars are in closer accord with a theory which is finite than with others which have not this property. The finiteness which can be demonstrated is a property of the theory and not of the universe. By a finite theory we mean one, such as the Riemannian geometry described in a preceding section, in which the distances of all points from any given fixed point are finite. The geometrical theory of Euclid has not this property.

6. *Mathematical theories in physics.* The physicist makes use of mathematics primarily as a correlator of heterogeneous data. One of the best examples of the possibility of using two slightly different theories for the correlation of the same physical phenomena lies in the relationship between Newtonian mechanics and the mechanics of the special theory of relativity. Every event in physical experience occurs at a particular time and place. If an observer A has a rectangular system of coordinates in space, and a mechanism for measuring time, he can locate each event by giving its time t and its three space coordinates x, y, z. Thus the location of an event requires the specification of four numbers t, x, y, z. This is all that the mathematical physicist means when he says that the world of events is a four-dimensional world.

If a second observer A' is moving with constant velocity v in the direction of the positive x-axis of the system of coordinates of A, and if A' has a clock and a

system of space coordinates parallel to those of A and moving with A', then each event will have four coordinates t', x', y', z' for A' as well as the four coordinates t, x, y, z of A. Newtonian mechanics tells us that these two sets of coordinates are related by equations of the form

$$t' = t, \quad x' = x - vt, \quad y' = y, \quad z' = z. \tag{1}$$

The special theory of relativity asserts, on the other hand, that these relations should be replaced by the equations

$$t' = \beta(t - vx/c^2), \quad x' = \beta(x - vt), \quad y' = y, \quad z' = z, \tag{2}$$

in which c is the velocity of light and β is merely a convenient symbol for the more complicated expression $\beta = 1/(1-v^2/c^2)^{1/2}$. It is unnecessary here to explain in detail the derivation of either of these sets of equations. For both theories they enable us to calculate the coordinates t', x', y', z' of an event when the coordinates t, x, y, z are given, or vice versa. Such relations are, of course, necessary if the world of events as observed by one of A and A' is to have any meaning to the other. Since the velocity of light in kilometers per second is the very large number $c = 300{,}000$ it is evident that for velocities v not exceeding a few kilometers per second the ratio v/c is exceedingly small and the value of β is very near to unity, so that the formulas (2) are almost identical with formulas (1). Thus for all moving bodies which we ordinarily consider on the surface of the earth, or for the motions of the planets, the two sets of equations give sets of values t', x', y', z' which are experimentally indistinguishable. The velocities of electrons in a current passing through a vacuum tube may, however, be a large fraction of c. For such cases the formulas (2) have been found to give results which agree with observations much better than those given by equations (1). Here, then, are two theories whose results differ by less than the errors of observation for every-day velocities. But one of them has a much better fit with observed data than the other when the range of velocities is enlarged.

The equations (2) are fundamental in the deduction of many of the results of special relativity theory. From them we can prove as mathematical theorems the often quoted assertions that according to the special theory of relativity a clock runs slower when it is moving than when it is stationary, that a yard-stick is shortened when moving in the direction of its length, or that there exist events occurring in one time order for A which appear to A' to take place in the opposite order. We need not worry about these matters from the practical standpoint, for the retardation of the clock and the shortening of the yard-stick at any velocities which we can experimentally impose upon them are too small to be measured by any apparatus which we now possess, and events whose orders are inverted must be so near together as to be practically indistinguishable.

As a final example of applied mathematical theories in physics one should mention the quantum theory. It is perhaps the most recently developed of all mathematical-physical theories, and is still undergoing modification and improvement. When an electric current consisting of electrons is passed through rarified

hydrogen gas in a vacuum tube, the atoms of the gas are excited and give off radiation which has a very definite and well-known line spectrum. One of the fundamental problems of the quantum theory is the determination of the character of the atoms of hydrogen and the mechanism by means of which they can give off the observed radiation. In the Bohr theory the atom is a tiny solar system consisting of a nucleus and an electron moving about each other in accord with a modified Newtonian theory of mechanics. By Newtonian mechanics the positions and velocities of the particles constituting the atom are determined uniquely at all times t when their positions and velocities at one time are known. This fact is a mathematical expression of the so-called principle of cause and effect. A second theory, matrix mechanics, provides for each time t only percentages of probability that the particles will be in various positions or have various velocities. According to this theory we would conclude that among the myriad of atoms of hydrogen in the tube a certain percentage have their particles in one position at a given time t, a second percentage have their particles in another, and so on. In some cases one of the percentages may be 100, or very nearly 100, and then we know with great precision what to expect. But in no case can a probability for positions and a probability for velocities be simultaneously 100. Thus we can never expect to have accurate information about positions and velocities at the same time. This, very roughly, is a description of the now famous "principle of uncertainty" of matrix mechanics, as contrasted to the principle of cause and effect of the older Newtonian theory. The principle of uncertainty has caused much discussion among philosophers and physicists since it seems to affect fundamentally our conception of the universe of phenomena in which we find ourselves immersed. If we accept the point of view of the preceding paragraphs, however, we must agree that the universe itself, sparsely interpreted for us by disconnected data, is not affected by either of these theories. Neither the principle of cause and effect, nor the principle of uncertainty, can be precisely characteristic of the behavior of nature. They are merely most interesting theorems in two different theories by means of which we endeavor to correlate and interpret observed data. The ultimate choice between the two must be determined by convenience or by their relative accuracies of fit with observation, and not because of any supposedly precise correspondence with nature on the part of either one of them.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The Department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A RECTANGULAR HYPERBOLA THEOREM

By J. R. MUSSELMAN, Western Reserve University

By proper choice of axes and scale the equation of any rectangular hyperbola

in Cartesian coordinates can be written parametrically as

$$x = t, \qquad y = t^{-1}. \tag{1}$$

The equation of the chord joining two points P_1 and P_2, whose parameters are t_1 and t_2 respectively, is

$$x + t_1 t_2 y = t_1 + t_2. \tag{2}$$

Given any three connected chords of a rectangular hyperbola P_1P_2, P_2P_3, P_3P_4; if one draws P_4P_5 and P_5P_6 parallel respectively to P_1P_2 and P_2P_3, then P_6P_1 is parallel to P_3P_4.

Due to the assumptions we have

$$\begin{aligned} t_1 t_2 &= t_4 t_5 \\ t_5 t_6 &= t_2 t_3. \end{aligned} \tag{3}$$

Upon multiplication of these two equations we have

$$t_1 t_6 = t_3 t_4 \tag{4}$$

and the theorem follows. We are then led to: *given any six connected chords of a rectangular hyperbola, so drawn that the last three chords are parallel respectively in order to the first three chords, the figure closes.* An interesting case occurs if the chords form either all positive, or all negative, 60° angles with each other. We then have an inscribed equiangular hexagon with opposite sides parallel. *Theorem: Given any n connected chords (n odd) of a rectangular hyperbola, if one draws $n-1$ more chords parallel respectively to the first $n-1$ chords, then $P_{2n}P_1$ is parallel to P_nP_{n+1}.*

From the assumptions it follows that

$$\begin{aligned} t_1 t_2 &= t_{n+1} t_{n+2} \\ t_{n+2} t_{n+3} &= t_2 t_3 \\ &\cdot \qquad\qquad \cdot \\ &\cdot \qquad\qquad \cdot \\ &\cdot \qquad\qquad \cdot \\ t_{n-2} t_{n-1} &= t_{2n-2} t_{2n-1} \\ t_{2n-1} t_{2n} &= t_{n-1} t_n. \end{aligned} \tag{5}$$

Upon multiplication of the above equations we have

$$t_1 t_{2n} = t_{n+1} t_n \tag{6}$$

and the theorem is proved. Hence it follows that *given $2n$ connected chords (n odd) of a rectangular hyperbola, so drawn that the last n chords are parallel respectively in order to the first n chords, the figure closes.*

If, in equation (1) we think of x and y as conjugate complex variables and t as a turn, i.e. $|t| = 1$; the algebra above shows that the theorems stated for chords of a rectangular hyperbola are also true for chords of a circle.

An Operational Formula

By C. A. Hutchinson, University of Colorado

In the discussion of linear ordinary differential equations with constant coefficients, the formula

$$\frac{1}{F(D^2)} \sin ax = \frac{1}{F(-a^2)} \sin ax, \quad F(-a^2) \neq 0,$$

is usually given for determining a particular integral of

$$F(D)y = f(x),$$

when $f(x) = \sin ax$. [$F(D)$ is a polynomial in $D \equiv d/dx$]. When $F(D)$ contains odd powers of D, some further manipulation is necessary, which is unconvincing to some students. To cover this case, I have derived the formula below, which proves very satisfactory to use. Let the zeroes of $F(D)$ be r_i, $i = 1, 2, \cdots, n$ (multiple zeroes enumerated individually). Then

$$\begin{aligned}\frac{1}{D - r_1} \sin ax &= e^{r_1 x} \int e^{-r_1 x} \sin ax \, dx \\ &= \frac{-r_1 \sin ax - a \cos ax}{a^2 + r_1^2} \\ &= \frac{(-D - r_1) \sin ax}{(ai - r_1)(-ai - r_1)}, \quad a^2 \neq -r_1^2.\end{aligned}$$

By repeated application

$$\begin{aligned}\frac{1}{F(D)} \sin ax &= \frac{1}{(D - r_n) \cdots (D - r_1)} \sin ax \\ &= \frac{(-D - r_n) \cdots (-D - r_1) \sin ax}{(ai - r_n) \cdots (ai - r_1)(-ai - r_n) \cdots (-ai - r_1)} \\ &= \frac{F(-D) \sin ax}{F(ai)F(-ai)}\end{aligned}$$

provided $F(ai)F(-ai) \neq 0$. If the coefficients of F are real, the last restriction may, of course, be replaced by $F(ai) \neq 0$. In this case, also, $F(-ai)$ is the conjugate of $F(ai)$. The formula for

$$\frac{1}{F(D)} \cos ax = \frac{F(-D) \cos ax}{F(ai)F(-ai)}, \quad F(ai)F(-ai) \neq 0,$$

is obtained in the same way.

For the hyperbolic functions the formulas are similar:

$$\left.\begin{aligned}\frac{1}{F(D)}\sinh ax &= \frac{F(-D)\sinh ax}{F(a)F(-a)}\\ \frac{1}{F(D)}\cosh ax &= \frac{F(-D)\cosh ax}{F(a)F(-a)}\end{aligned}\right\} F(a)F(-a)\neq 0.$$

NOTE ON THE LAPLACIAN OF A VECTOR POINT FUNCTION

By J. J. SLADE JR., Rutgers University

In most of our text books on vector analysis there appears, without comment, the formula

$$\nabla^2 u = \text{grad div } u - \text{curl curl } u \tag{1}$$

where u is the vector

$$u_1(x, y, z),\ u_2(x, y, z),\ u_3(x, y, z)$$

and $\nabla^2 u$ is

$$\nabla^2 u_1,\ \nabla^2 u_2,\ \nabla^2 u_3.$$

This formula is quite misleading, for whereas the right hand member of (1) is invariant for all the transformations of classical physical space, the left hand member is not.

For let u be transformed into v by the substitution

$$T:\begin{cases} x = x(\xi, \eta, \zeta)\\ y = y(\xi, \eta, \zeta)\\ z = z(\xi, \eta, \zeta)\end{cases}$$

which is supposed to be orthogonal in the sense that

$$ds^2 = \left(\frac{d\xi}{h_1}\right)^2 + \left(\frac{d\eta}{h_2}\right)^2 + \left(\frac{d\zeta}{h_3}\right)^2.$$

The divergence and curl of u are transformed into the divergence and curl of v, and the gradient of a scalar point function ϕ goes into the gradient of $T\phi$. But, if we let $\overline{\nabla}^2$ stand for the operator

$$h_1h_2h_3\left\{\frac{\partial}{\partial\xi}\frac{h_1}{h_2h_3}\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\frac{h_2}{h_3h_1}\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\zeta}\frac{h_3}{h_1h_2}\frac{\partial}{\partial\zeta}\right\}$$

we see that

$$\nabla^2 u \to \overline{\nabla}^2\left[\sum v_ih_i\frac{\partial x}{\partial\xi},\ \sum v_ih_i\frac{\partial y}{\partial\xi},\ \sum v_ih_i\frac{\partial z}{\partial\xi}\right] \tag{2}$$

which is never the same as $\overline{\nabla}^2 v$ except for the trivial case when T is the identity transformation.

If u satisfies Laplace's equation, $\nabla^2 u = 0$, then each of the components of the right-hand member of (2) vanishes. Expanding the first of these we get

$$(3) \qquad \sum v_i \overline{\nabla}^2 \left(h_i \frac{\partial x}{\partial \xi} \right) + 2 \sum \overline{\nabla} v_i \cdot \overline{\nabla} \left(h_i \frac{\partial x}{\partial \xi} \right) + \sum h_i \frac{\partial x}{\partial \xi} \overline{\nabla}^2 v_i = 0,$$

where $\overline{\nabla}$ stands for the operator

$$h_1 \frac{\partial}{\partial \xi}, \quad h_2 \frac{\partial}{\partial \eta}, \quad h_3 \frac{\partial}{\partial \zeta}.$$

From this it is seen that, unless there is a functional relation between v and the coördinate elements, the vanishing of $\overline{\nabla}^2 v$ implies the vanishing of $\overline{\nabla}(h_i \partial x/\partial \xi)$; that is, $h_i \partial x/\partial \xi = \text{const.}$ and the transformation T is a linear orthogonal transformation. Conversely, if T is linear, then $\overline{\nabla}(h_i \partial x/\partial \xi) = 0$, $\nabla^2 (h_i \partial x/\partial \xi) = 0$ and $\sum h_i (\partial x/\partial \xi) \overline{\nabla}^2 v_i = 0$ which implies that $\overline{\nabla}^2 v_i = 0$, $i = 1, 2, 3$; that is, $\overline{\nabla}^2 v = 0$.

In other words, although, when T is not the identity transformation, $\nabla^2 u$ never transforms into $\overline{\nabla}^2 v$, $\nabla^2 u = 0$ implies $\overline{\nabla}^2 v = 0$ when and only when T is a linear orthogonal transformation.

Formula (1) has appeared in the literature in a rather artificial manner; it is a statement limited to a vector point function referred to a fixed rectangular cartesian frame; it is a fairly interesting relation, but should not in my opinion be included among the relations of vector analysis.

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Algebra. By Oskar Perron. Second Edition. Berlin and Leipzig, Walter de Gruyter and Co. Volume I, Die Grundlagen, 1932. viii+301 pages. Volume II, Theorie der Algebraischen Gleichungen, 1933. viii+261 pages.

The first edition of this two-volume work, which has for several years had a recognized place in our libraries, was distinguished by the prominence given to the field concept. In the new edition there is an improved presentation of this concept and its fundamental significance in many parts of the theory is further emphasized. The author retains as his objective the desire to write a book available to a beginner, but one which takes account of recent research in the subject.

In the first chapter of volume one an abstract definition of *field* is given and discussed before it is specialized to the number field definition used in the earlier edition. Here, as well as in many other places where fundamental ideas are presented, developments which are not essential but which are confusing to the

novice are excluded; only fields of characteristic zero and rings with units and no divisors of zero are discussed. The complex field is obtained by use of the Hamilton number pair definition, a treatment which finds satisfactory generalization in a later chapter where the König procedure is employed to set up a field which contains a root of a given equation in one variable irreducible in a given field. This treatment leads to a discussion of the field of decomposition of any polynomial in one variable, the results of which are applied in the proof of the so-called "Fundamental Theorem of Algebra," where the older Gauss-Gordan proof has been replaced by the Artin-Schreier proof of 1927 (given also by Dörge in 1928). This proof has the merit of greater brevity and is an elegant application of the field theory already mentioned.

Consistent with the change in the introduction of the field notion is the change in the treatment of groups in the second volume. The definition of an abstract group is given at the outset and the isomorphism of a finite abstract group with a substitution group proved. This makes possible a simpler and more satisfactory treatment of factor groups than appeared in the first edition. Throughout the second volume the author has achieved considerable simplification in the proofs of his theorems and greater clarity in their statement.

A section on number theory has been inserted before the discussion of linear groups; the subject is presented in such a way that it grows naturally out of the group notions already discussed and gives a satisfactory preparation for the linear group theory which follows.

After the subject of finite extensions of a field has been treated, there is an excellent exposition of the Galois group, co-ordinating the presentation given in the first edition with the more recent procedure based on the automorphisms of the field of decomposition of the given equation. The notion of automorphisms, however, is not used in subsequent proofs; the author retains the proofs based on his earlier definitions. But it must be said that a student who has read and understood this treatment should have no difficulty in understanding a presentation of the subject such as the one given by van der Waerden.

Chapter VII of the first volume has been enlarged to include additional material on systems of equations and the conditions for the existence of solutions, and a new and improved proof of Bézout's theorem which takes account of the multiplicity of solutions. In the second volume some sections on orthogonal transformations and characteristic equations have been added, but, as in the first edition, certain familiar topics such as elementary divisors and pairs of bilinear forms are not mentioned.

Insistence on clarity and concreteness in the presentation of details make this a good text for beginning students, though the more experienced reader may feel at times that the author has been overscrupulous in the inclusion of all details. Professor Perron has succeeded admirably in preserving the high standard of the first edition, while making revisions which should enable the student to read with greater ease current algebraic research.

MINA S. REES

Elementary Mathematical Analysis. By Mayme Irwin Logsdon. New York, McGraw-Hill Book Company. Vol. I (1932) xiv+212 pages, 5 tables. $2.25. Vol. II (1933) ix+188 pages. $1.75.

These two volumes represent an attempt to incorporate into a more or less harmonious whole the topics usually treated in the courses preparing for Calculus, namely, Trigonometry, Algebra, and Analytics. In order to achieve this harmonious integration the concept of function is introduced at the outset. This naturally leads to the study of the derivative and its application to the elementary functions. These ideas are the only ones that are foreign to the usual courses in these subjects. This, of course, in the words of the author, makes for "simplicity and continuity" although it does rearrange somewhat the order of the topics.

Roughly, the order of discussion is that mentioned above, although some topics found in the traditional course in Algebra have been placed at the end of Volume II because they do not fit into the logical treatment that is followed. Each subject receives approximately the same number of pages.

Review in the technique acquired in high school mathematics is effectively presented. For example, the theory of exponents is recalled in the chapter on logarithms and restated in terms of logarithms. This plan has been followed throughout the books, that is, a technique is reviewed at the time it is needed.

As for Trigonometry, the usual material is given. The functions are defined for the general angle in the beginning, thereby effecting a considerable saving of time and space. In the treatment of oblique triangles only the cosine law is made available for the case of three sides given. Would it not be desirable to have an adequate table of squares included so as to place this case on a more equal footing, as regards accuracy and simplicity of solution, with the other cases? A short chapter on the history of the development of Trigonometry is placed just before the solution of triangles. A feature that will find favor with the student is that the pronunciation of Greek letters is given when they are first introduced. Did the reviewer overlook the statement that τ is tau (I, p. 162)?

The discussion of the number system of elementary mathematics (II, Chap. 1) is excellent and will gratify those instructors who have been in the habit of supplying such material as a preliminary to the study of complex numbers. There is a paragraph (II, p. 149) to help the student understand when mathematical induction is needed for the proof of a formula and when not. May it remove some of the terrors of that topic!

A feature that, fortunately, one finds more and more in texts on analytics, is stressed in these volumes, namely, the close correspondence between ideas, elements, and language of geometry and those of algebra. (I, p. 9, II, p. 51). In fact, whenever facts can be expressed from two points of view, these statements are exhibited in parallel columns (I, p. 68, II, p. 122, for example). The treatment of conics varies somewhat from that to which one is accustomed. For example, it is stated that an ellipse is the graph of a certain equation. Then

the string property is stated and the equation of this locus is shown to be the same as the one first given. Many may object to the fact that there are few, if any, locus problems. The only discussion of this fundamental problem that the reviewer noticed, in this connection, is in Volume II, p. 98 where the equation of the general conic is derived from its definition as a locus. The author feels constrained to defend her short treatment of translation of axes. The writer believes that the insertion of exercises (there are none) at this point would be of assistance to the student.

Each chapter is opened by a paragraph stating what is to be accomplished in the chapter and at the end a short summary of the contents is listed. For quick reference the chapter and section numbers are printed at the head of each page. The two volumes afford a comprehensive analysis of the polynomial, rational, trigonometric, logarithmic, and exponential functions. In fact, an ϵ-definition of a continuous function is given (I, p. 24) and certain of the above functions are *proved* to be continuous!

The two volumes have been designed to give a sound preparation for a course in Calculus and require four hours per week for a year, or, three hours per week for a year and a half. The reviewer believes that they would serve for a year's work at three hours weekly, provided one is willing to postpone to a later course such chapters as those on determinants and the simultaneous solution of three linear equations, pencils of curves, etc.

We list here the few typographical errors that have come to notice:

Vol. I, p. 17, line 16, for sam read same.
Vol. II, p. 53, bottom of page, for concurrent read collinear.
Vol. II, p. 112, line 10, omit parentheses.

C. A. NELSON

Knotentheorie. Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 1, No. 1. By K. Reidemeister. Berlin, Julius Springer, 1932. vi+74 pages. RM 8.75.

This is the first of a series of monographs sponsored by the editors of the Zentralblatt fuer Mathematik. In a preface to the series they express the belief that these monographs will supplement the work of the Zentralblatt by strengthening and consolidating the positions attained by individual research on a particular topic. It is expected that the series will supplement also the Encyklopädie by giving the more recent results in each special field in the form of monographs which will be largely complete in themselves. This first member of the series adheres very closely to these ideals. There is a detailed and careful exposition of the modern theory of knots and their projections, invariants and groups. At the end of the monograph there is an excellent bibliography of the literature, and there are frequent references to this in the text so that the reader may orientate himself in the literature of knots. There is also appended the table of knots due to Alexander and Briggs.

W. L. AYRES

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscripts should be typewritten, with double spacing, and with margins at least one inch wide.

CLUB ACTIVITIES

1932–33

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are engaged in study and research and the exchange of ideas in the field of mathematical science.

Pi Mu Epsilon of the University of California

The officers for the coming year are: Mr. John Oxtoby, Director; Mr. Raphael M. Robinson, Vice Director; Miss Myra Waddell, Secretary; Mr. Willis E. Lamb, Jr., Treasurer; Mr. Gabriel G. Bejarano, Librarian. These officers were elected by a majority vote of the members present at a regular meeting of the society.

There are fifty active members in the society, twelve were initiated on October 8, 1932, and seventeen on March 4, 1933.

The meetings and programs were as follows:

August 24, 1932: "Polygenic functions" by Mr. William Hutchings.
September 21, 1932: "The orbit problem" by Dr. Sophia Levy.
October 19, 1932: "Our journey through space" by Miss Phyllis Hayford.
November 16, 1932: "Hypergeometry" by Dr. Wong.
January 25, 1933: "Vector analysis" by Mr. Floyd Fisher.
February 15, 1933: "Transfinite numbers" by Mr. Raphael Robinson.
March 16, 1933: "Number ratios" by Mr. John Oxtoby.
April 16, 1933: "Coordinate systems" by Mr. Everett Matthews.

Our annual picnic was held at Cordonices Park on May 5, 1933, and despite the fact that this was the last week of finals it was well attended.

MYRA WADDELL, *Secretary*

Pi Mu Epsilon of the University of California at Los Angeles

We are very glad to be able to report a most successful year under the direction of the following officers elected in the fall of 1932: Caroline Dutton, Director; Ruth Cunningham, Vice Director; Walter Roberts, Secretary; Wendell Mason, Treasurer; Carroll Brady, Librarian.

The director, the vice director and the secretary are undergraduate students. We have at the present time an active membership of thirty-five and a faculty membership of fourteen. Ten students were admitted to membership in the fall and six in the spring.

During the eight regular monthly meetings, the following activities were enjoyed:

October 10, 1932: "Some analogies between algebraic and differential equations" by Mr. D. Hyers.
November 9, 1932: Informal introduction of pledges.
December 7, 1932: "Related difference and differential equations" by Miss Bordon; "Applications and tricks of integration" by Mr. Garver.
January 11, 1933: "The series method in differential equations" by Mr. Roberts; "Thomas Jefferson and his mathematics today" by Miss Glazier.
February 15, 1933: The presentation and discussion of revised By-Laws.

April 12, 1933: "Quadratic involutions" by Miss Cunningham; "Calculation with a mathematical computer" by Mr. Daus.
May 17, 1933: "Tensor analysis" by Mr. Richardson.
June 7, 1933: Election of officers for 1933–1934.

The Fall initiation was held November 19, 1932 at the home of Mr. and Mrs. Wendell Mason. The Spring initiation, a semi-formal banquet, took place on May 6, 1933 at the Miramar Beach Club in Santa Monica. At Christmas an informal party was held in conjunction with the University Mathematics Club.

Every year our organization offers a prize calculus examination. This year it was held on May 16, 1933. Mr. D. Kalbfell and Mr. Ralph Phillips tied for highest honors. Honorable mention was awarded to Miss Sarah Bordon. We are also pleased to announce the election of three of our members to Phi Beta Kappa. They are Elizabeth Breuer, Donald Hyers, Reginald Richardson.

WALTER ROBERTS, *Secretary*

Pi Mu Epsilon of Hunter College of the City of New York

The Hunter College Chapter of Pi Mu Epsilon devoted the four program meetings of the fall term to the topic: "Invariants and Covariants." Attention was restricted to the fundamental processes of invariant theory and to the concomitants of the binary quadratic and cubic. Nine students reported on this topic. During the second term eleven students read papers selected from "Higher plane curves" and the "Calculus of finite differences."

Twenty-seven students and two faculty members were initiated during the year, bringing the active membership up to fifty-three.

In November the fraternity held its formal initiation at a dinner party at the Hotel Barbizon. Sixty-three active and alumni members attended. The chapter was especially honored on this occasion by the presence of Professor Edward Kasner as guest speaker. The second initiation of the year followed by a tea and bridge party was held in March.

FRIEDA ZWECHER, *Corresponding Secretary*

LOCAL MATHEMATICS CLUBS

The Mathematics Clubs of Carleton College

At Carleton College there are two organizations of individuals interested in mathematics. Their activities are more or less interrelated. First, there is the School of Crotona with membership limited to those who have passed a comprehensive oral examination in mathematics through integral calculus. Second, there is the Mathematics Club with unlimited membership. The School of Crotona serves as the executive council of the latter since the members of this smaller group conduct the programs and select the officers of the Mathematics Club.

At present, William Watson is President of the Mathematics Club; Philip Nason is Secretary-Treasurer. These officers were elected by the members of the School of Crotona. There are usually eight active members in the School of Crotona. The attendence at the meetings of the Mathematics Club varied from about twenty to forty.

The primary aim of the organizations is to combine a student discussion of mathematics with coffee, tea, and rolls—the Greek idea of congenial intellectual discussion.

Six meetings were held this year. They were as follows:
October 14, 1932: "The cocoanut problem" by William Watson.
November 7, 1932: "Mathematical problems and puzzles" by William Lee.
December 7, 1932: "Solutions of quadratic equations" by Edward Tomastic.
January 18, 1933: "Probability" by Philip Nason.
February 20, 1933: "The use of mathematics in objective testing" by Dr. Selmer Larson.
May 4, 1933: "Trisection of angles and other mathematical problems and puzzles" by Ray Wendland, Raymond Jurgensen, and Selma Kjontvedt.

WILLIAM WATSON, *President*

The Mathematics Club of Butler University

The officers for the academic year 1932–1933, elected by secret ballot in May 1932, were: Rhom Settles, President; Thelma Tacoma, Vice President; Douglas Ewing, Treasurer; Fletcher Rahke, Secretary.

The purpose of the club is two-fold: to consider and discuss historic and current mathematical topics and to give opportunity to all students in the department to become acquainted.

Membership in the club is granted to all students in the department of mathematics, to those who have been students in the department, and to sponsors. The active membership numbered twenty-two for the year 1932–1933.

The meetings and programs for the year were as follows:

October 6, 1932: "The organization and history of the club" by Douglas Ewing; "Mathematics and music" by Thelma Tacoma.

November 3, 1932: "Magic squares" by Anna K. Suter, Assistant in the department of mathematics.

December 8, 1932: Christmas party, mathematical games and puzzles.

January 13, 1933: The reunion and anniversary meeting. Forty-five persons attended. At this meeting, Dr. S. E. Elliott, Professor of Physics of Butler University discussed "The Del-operator."

February 2, 1933: "The Einstein theory of relativity" by Professor W. D. Carnahan, Head of the department of mathematics at Shortridge High School.

March 2, 1933: "The trisection of an angle" by Miss Florence Rathart (read by John Batcheler); "Professor Callahan's attempt at the trisection of an angle" by Miss Donnabelle Naylor.

April 6, 1933: "The science of mathematics, its origin, development and importance" by Mr. Walter Gingery, Principal of Washington High School.

May 11, 1933: "A trip through space" by Panoria Apostol (read by Miss Harriet Summers).

June 2, 1933: Annual picnic on the Butler campus.

FLETCHER T. RAHKE, *Secretary*

The Mathematics Club of Adelphi College

The officers of our mathematics club were: Dorothy Hill, President; Hazel Geis, Vice President; Clara Strein, Secretary; Doris Bettman, Treasurer; Dr. Joseph Bowden, Honorary President. The annual election of officers occurs at the May meeting. There are approximately thirty active members.

The primary aim of the club is to further interest in mathematics. Membership in the club is open to all students interested in mathematics.

The meetings and programs were as follows:

September 27, 1932: Business meeting.

October 17, 1932: Social meeting. A tea was held to welcome the new members into the club.

October 31, 1932: "Life of Sir Isaac Newton" by Miss Doris Bettman; "Life of Sir Christopher Rene" by Miss Isabel Hill.

November 21, 1932: "Generalized coordinates" by Miss Leon, a member of the faculty.

December 6, 1932: "Discussion of Teacher-In-Training Examination" by Miss Gordon, a 1932 graduate.

January 17, 1933: "Scales of notation" by Dr. Bowden, Honorary President.

February 20, 1933: "Recent methods in the teaching of mathematics" by Miss Flora D'Amato; "Primitive methods of counting" by Miss Hazel Geis.

March 16, 1933: Joint meeting of the Chemistry and Mathematics clubs.

April 3, 1933: "Relation of mathematics to astronomy" by Professor David, Professor of Physics and Astronomy.

May 1, 1933: "Scales of notation" by Dr. Bowden.

May 9, 1933: Special business meeting; nomination of officers for the year 1933–1934.

May 16, 1933: Special business meeting; election of officers for the year 1933–1934.

May 22, 1933: Special meeting; installation of officers.

CLARA STREIN, *Secretary*

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to Wm. Fitch Cheney, Jr. Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 54. *Proposed by P. R. Hill, University of Georgia.*

If the probability that an event will occur in a single trial is $1/N$, then as N increases without limit the probability that the event will occur at least once in N trials approaches the limit, $1-1/e$.

E 55. *Proposed by J. Rosenbaum, Milford, Connecticut.*

Obtain the general solution in positive integers of

$$2z^3 = x + (x^2 - 4y^3)^{1/2}.$$

E 56. *Proposed by Otto Dunkel, Washington University, Saint Louis, Missouri.*

From the base vertices A and B of an isosceles triangle ABC, segments of straight lines AL and BM of equal length are drawn to the opposite equal sides. Determine by plane geometry the locus of P, the intersection of AL and BM.

E 57. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

Tom, Dick and Harry were comparing notes on their fishing experiences of the previous summer, and found that one had caught only perch, one only trout, and one only salmon. They remarked that the number of fish Tom had caught was seven more than three-fifths of the number of perch that were caught; that the number of fish Dick had caught was three more than five-sevenths of the number of salmon that were caught; and that the total number of all the fish caught was a three-place prime number. Determine how many and what kind of fish each caught, and show that the solution is unique.

E 58. *Proposed by R. M. Sutton, Haverford College, Pa.*

In the following division of a three-place number into a five-place number each digit has been replaced by a code letter. Assuming only that the remainder, Y, is not zero, reconstruct the problem and show that the solution is unique.

$$\begin{array}{rr} L\ M\ N) & R\ S\ T\ U\ N(U\ X \\ & \underline{R\ T\ Y\ X\ \ } \\ & \ \ T\ Y\ Y\ N \\ & \ \ T\ Y\ Y\ J \\ & \underline{\ \ \ \ \ \ \ Y} \end{array}$$

E 59. *Proposed by J. H. Butchart, Indianapolis, Indiana.*

In the angle ACB of triangle ABC circles are inscribed tangent respectively to AC at A and to BC to B. Prove that the chords intercepted on the side AB are equal.

SOLUTIONS

E 26. [1933, 175] *Proposed by H. T. R. Aude, Colgate University.*

All proper rational fractions in lowest terms can be separated into two classes:

1. Those in which the numerator and denominator are both odd.

2. Those in which the numerator and denominator are not both odd. Show that for any fraction F in one class there is just one corresponding fraction F' in the other such that $\arctan F+\arctan F'=\pi/4$.

Solution by A. E. Andersen, Wagner College, Staten Island, N. Y.

Since $(F+F')/(1-FF')=\tan \pi/4=1$, we have $F'=(1-F)/(1+F)$.

If F belongs to class 1, it may be written in the form $(2n+1)/(2m+1)$, where both m and n are positive integers. Substitution and reduction gives $F'=(m-n)/(m+n+1)$, and since this numerator and denominator differ by the odd number, $2n+1$, one must be odd and the other even, so that F' belongs to class 2.

If, on the other hand, F belongs to class 2, it may be written in one of the two forms, $2n/(2m+1)$ or $(2n+1)/2m$. In these cases substitution shows $F'=(2m-2n+1)/(2m+2n+1)$ or $(2m-2n-1)/(2m+2n+1)$, each of which forms is of class 1.

Solved also by L. S. Johnston, Theodore Lindquist, J. Rosenbaum, Simon Vatriquant and R. N. Walter.

E 27. [1933, 175] *Proposed by E. P. Starke, Rutgers University.*

Derive the algebraic formula for the sum of n fractions whose numerators are in arithmetic progression and whose denominators are in geometric progression.

Solution by H. E. H. Greenleaf, De Pauw University, Greencastle, Indiana.

Let the sum of the fractions be

$$S=\frac{a}{b}+\frac{a+d}{br}+\frac{a+2d}{br^2}+\frac{a+3d}{br^3}+\cdots+\frac{a+(n-1)d}{br^{n-1}}$$

$$=\frac{a[1+r+r^2+\cdots+r^{n-1}]+d[(n-1)+r(n-2)+r^2(n-3)+\cdots+2r^{n-3}+r^{n-2}]}{br^{n-1}}.$$

The coefficient of a is seen to be $(1-r^n)/(1-r)$.

The coefficient of d may be written with each power of r copied as many times as called for by its coefficient, thus:

$$
\begin{aligned}
&1\\
&+1+r\\
&+1+r+r^2\\
&+1+r+r^2+r^3\\
&+\cdots\cdots\cdots\cdots\\
&\quad\cdots\cdots\cdots\cdots\\
&+1+r+r^2+\cdots+r^{n-3}+r^{n-2}
\end{aligned}
$$

for $n-1$ rows.

If this is summed horizontally and the results added, the coefficient of d reduces to $(1-r+1-r^2+1-r^3+\cdots+1-r^{n-1})/(1-r)$, or

$$[n(1-r)-(1-r^n)]/(1-r)^2,$$

so that finally

$$S=\frac{(a-ar-d)(1-r^n)+nd(1-r)}{br^{n-1}(1-r)^2}.$$

In their solutions of this problem, F. C. Gentry and R. MacKay point out that the series of fractions converges if r exceeds unity, and that if n then increases without limit, the limit of S is

$$\frac{ar^2+r(d-a)}{b(r-1)^2}.$$

R. MacKay and R. N. Walter remark that this problem is very closely related to one on page 44 of Hall and Knight's *Higher Algebra*.

Solved also by Martha Allen, A. E. Andersen, W. E. Buker, L. S. Johnston, Theodore Lindquist, F. L. Manning, Lazerus Medveson, Jr., Charles Molloy, W. R. Ransom, C. C. Richtmeyer, J. Rosenbaum, H. D. Ruderman, Simon Vatriquant and someone who forgot to sign his solution.

E 29. [1933, 175] *Proposed by J. Rosenbaum, Milford, Connecticut.*

The faces of a tetrahedron are congruent triangles whose sides are a, b and c. If $2S=a^2+b^2+c^2$, show that the volume is $V=[(S-a^2)(S-b^2)(S-c^2)]^{1/2}/3$.

Solution by Simon Vatriquant, Athénée Royale d'Ixelles, Brussels, Belgium.

Denoting by A, B and C the angles opposite the sides a, b and c in each of the four triangles, the area of each triangle is $K=\frac{1}{2}ab\sin C$.

The altitude h of the tetrahedron is $c\sin H$, where H is the angle between the edge c of the trihedral angle at a vertex of the tetrahedron and the opposite

face of that trihedral angle. Cutting this trihedral angle by a sphere centered at its vertex gives a spherical triangle of sides A, B and C, in which H is the arc-altitude to the side C. In this spherical triangle, $\sin H = \sin A \sin P$, where P is the angle of the spherical triangle opposite the side B. Finally P is given in terms of the sides of the triangle by the formula

$$\sin P = 2[\sin \tfrac{1}{2}(A + B + C) \sin \tfrac{1}{2}(B + C - A) \sin \tfrac{1}{2}(C + A - B) \sin \tfrac{1}{2}(A + B - C)]^{1/2}/\sin A \sin C.$$

But $A+B+C=180°$, so that this formula reduces to

$$\sin P = 2[\cos A \cos B \cos C]^{1/2}/\sin A \sin C.$$

Then the altitude h of the tetrahedron is $2c[\cos A \cos B \cos C]^{1/2}/\sin C$, and the volume $V = Kh/3 = abc[\cos A \cos B \cos C]^{1/2}/3$.

Now from the cosine law for a plane triangle we may readily derive $ab \cos C = \frac{1}{2}(a^2+b^2-c^2) = S-c^2$, and the two similar relations. When we use these relations to eliminate the cosines from the formula just derived for the volume, there results the desired formula, $V = [(S-a^2)(S-b^2)(S-c^2)]^{1/2}/3$.

Solved also by W. E. Buker, H. E. H. Greenleaf, R. N. Walter, the proposer and the anonymous solver of E 27.

E 30. [1933, 175] *Proposed by W. F. Cheney, Jr., Connecticut State College.*

The arithmetic mean between NED and $SASH$ is $SHUN$. Their geometric mean is $SEND$, and their harmonic mean is $SEED$. Assuming that the seven letters here involved represent different digits, identify them.

Solution by W. E. Buker, Leetsdale, Pa.

1. Since $NED+SASH=2(SHUN)$, S must be 1.

2. $(NED)(SASH)=(SEND)^2$. Thus $(D)(H)$ must equal a number whose right hand digit could be the right hand digit of D^2. The values of (D, H) for which this is possible are (2,7), (4,9), (8,3) and (5, any odd digit).

3. But from (1), $D+H$ must be even, so that D is 5 and H is 3, 7 or 9.

4. From (1), $SHUN<1500$, so that $H=3$.

5. Since from (1), $2(SHUN)$ now ends in 8, N is 4 or 9. But if N were 4, $SHUN$ could not equal $13xx$ because of (1), so N must be 9.

6. (2) has now become $(9E5)(1A13)=(1E95)^2$, so that $(9E5)(1A13)$ must end in 25. This forces E to be 2. Then $(925)(1A13)=(1295)^2$, and A is 8.

7. (1) has now become $925+1813=2(13U9)$, so U is 6.

8. Finally, $SEED$ is 1225, which is the harmonic mean of 925 and 1813.

It is interesting to note that the complete solution was obtained without using the given information concerning the harmonic mean.

Solved also by M. L. Constable, F. C. Gentry, F. L. Manning, W. R. Ransom, Elijah Swift, Simon Vatriquant, J. H. Weaver and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3631. *Proposed by Norman Anning, University of Michigan.*

On the Argand diagram a regular pentagram is constructed with its center at the origin and one vertex on the axis of positive real numbers. Find the equation whose roots are the numbers associated with the ten points where the lines intersect. Can the scale be so chosen that the equation will have integral coefficients?

3632. *Proposed by Morgan Ward, California Institute of Technology.*

If $r_1, r_2, \cdots, r_k$ are k fixed positive numbers whose sum is greater than one, and if

$$u_n = \sum_{(m_1+m_2+\cdots+m_k=n)} m_1^{-r_1} m_2^{-r_2} \cdots m_k^{-r_k},$$

where the summation variables m have the range $1, \cdots, n$, then, after a certain value n_0 of n, u_n decreases monotonically to zero with increasing n.

3633. *Proposed by N. A. Court, University of Oklahoma.*

If the three face angles of a trihedral angle of a tetrahedron are right angles, the line joining this vertex of the tetrahedron to the centroid of the opposite face is equal to two thirds of the circumradius of the tetrahedron.

3634. *Proposed by J. Rosenbaum, Milford, Connecticut.*

In the triangle ABC, the bisectors of the angles A and B meet the opposite sides at D and E. Prove that if DE divides the angles CDA and CEB into parts having equal ratios then the sides CA and CB are equal.

3635. *Proposed by Martin Rosenman, Brooklyn, New York.*

Given irrational numbers, x and y, and any positive quantity e, determine integers r, s, and t such that

(a)
$$0 < tx - r < e,$$
$$0 < ty - s < e.$$

(b)
$$0 < rx - t < e,$$
$$0 < sy - t < e.$$

3636. *Proposed by B. F. Finkel, Drury College.*

An equilateral triangular lamina of uniform density and of mass m is suspended in a horizontal position by three equal inextensible strings fastened to its three vertices A, B, C, and also to a point, P, above the lamina. A weight, w, is placed at any point Q inside the triangle ABC. Determine the tension in each string.

3637. *Proposed by Otto Dunkel, Washington University.*

Triangle ABC is isosceles with the vertical angle C greater than 60° and less than arc cos 1/3. Through a point P in its interior the segments of straight lines $AL=p$, $BM=q$, $CN=r$ are drawn to the opposite sides. Show that there are two points on its interior such that $p=q=r$, and that each of these segments has the length $3^{1/2}c/2$, where $c=AB$.

Show also that, adjacent to each of the equal sides $AC=BC$, there is in the interior of the triangle a region bounded by a part of that side and by parts of two curves which meet in one of the above two points, such that for each interior point r is the longest of the three segments. See problem 3576 [1932, 549].

3638. *Proposed by Oystein Ore, Yale University.*

Let p be a prime and N the smallest exponent such that $p^N \equiv 1 \pmod{n}$ for a given number n. The irreducible factors of $x^n-1 \pmod{p}$ then have degrees dividing N. Find the necessary and sufficient condition, that there exist prime divisors of $x^n-1 \pmod{p}$ of degree N' for all divisors N' of N.

SOLUTIONS

3575. [1932, 549] *Proposed by Frank Morley, Johns Hopkins University.*

Given two circles in a Euclidean space, which are not interlaced and not cospherical; show that there are four circles which touch both, and that these break into two pairs, the three pairs forming a symmetrical configuration.

Solution by N. A. Court, University of Oklahoma.

If a circle is tangent to the two given circles (a), (b), the point L common to the planes of the three circles has equal powers with respect to (a) and (b), or, what is the same thing, with respect to the two spheres (A), (B), having (a), (b), for diametral circles. Thus the point L is the trace of the radical plane σ of (A), (B), on the line of intersection m of the planes of (a), (b), and may therefore, in general, be determined.

Let P, Q; R, S; be the points of contact of the two pairs of tangents LP, LQ; LR, LS; from L to (a) and (b). The circle (PR) passing through P, R, and tangent to LP at P will also be tangent to LR at R; hence (PR) is tangent to both (a) and (b). Similarly, for the circles (PS), (QR), (QS).

Each of the six circles $(PQ)\equiv(a)$, $(RS)\equiv(b)$, (PR), (PS), (QR), (QS), is tangent to four of the remaining five circles. Since

$$LP = LQ = LR = LS,$$

the four points P, Q, R, S, lie on a sphere (L) having L for center. The planes of the six circles are the faces of the complete four-edge determined by the radii LP, LQ, LR, LS, of (L). The figure may thus be obtained by taking four points P, Q, R, S, on a given sphere (L), and it is immaterial how these four points are separated into two groups to obtain the first two circles (a), (b).

Special cases. (i). The plane σ passes through the line m. Every point of m may be made to play the role of the point L. The problem has an infinite number of solutions. However, this case is excluded by the wording of the problem, for it may be shown that the circles (a), (b), are cospherical in this case.

Indeed, since σ is perpendicular to the line of centers AB, and m, by hypothesis, lies in σ, the two lines AB, m, are orthogonal, and the plane through AB perpendicular to m contains the axes c, d, of (a), (b). Let (G), (G'), be the two spheres passing through (a), (b), respectively, and having for center the point $G \equiv cd$. Since m lies in σ, any point U of m is the center of a sphere (U) orthogonal to both (A) and (B). Now U lies in the radical plane of (A) and (G); hence (U) is also orthogonal to (G). Similarly, (U) is orthogonal to (G'). But with the point G as center only one sphere may be drawn orthogonal to (U); hence (G) and (G') are identical, i.e., the two circles (a), (b) will have to be co-spherical, which is contrary to the assumption of the problem.

(ii). If the plane σ is parallel to m, the points P, Q, R, S are the ends of the two diameters of (a), (b), which are perpendicular to m, and the solution does not differ from the general case. The axes c, d, have a point in common, but (a), (b), are not cospherical.

(iii). If the planes of (a), (b), are parallel, and the line AB is perpendicular to these planes, the two circles are co-spherical. If AB is not perpendicular to the planes of (a), (b), the points P, Q, R, S are the ends of the two diameters of (a), (b), which lie in the plane through AB perpendicular to the planes of (a), (b). The solution is the same as in the general case.

(iv). If the two circles (a), (b), are concentric, the same is true about the spheres (A), (B), and σ is at infinity, so that the point L coincides with the point at infinity of m, and the points P, Q; R, S, are the ends of the two diameters of (a), (b), perpendicular to m.

Note. The problem is proposed as an exercise in Jacques Hadamard's *Leçons de Géométrie Élémentaire*, Paris, 1901 vol. II, p. 149, ex. 689.

Also solved by Rufus Crane, C. A. Rupp and the proposer.

3579. [1932, 607] *Proposed by B. F. Kimball, Schenectady, N. Y.*

Given

$$F(x, a) = \sum_{s=-\infty}^{+\infty} \frac{\sin (x + sa)}{x + sa}$$

where s takes on all integral values including zero. Evaluate $F(x, a)$ for all real values of x and a. Discuss the discontinuities of the function.

Solution by the Proposer

Given

$$F(x, a) = \sum_{s=-\infty}^{+\infty} \frac{\sin (x + sa)}{x + sa}. \tag{1}$$

Expand $\sin z(\pi - \theta)$ in sine series in θ over interval $0 < \theta < 2\pi$.

$$\sin z(\pi - \theta) = \frac{\sin \pi z}{\pi} \sum_{s=-\infty}^{+\infty} \frac{\sin s\theta}{z + s}.$$

Similarly,

$$\cos z(\pi - \theta) = \frac{\sin \pi z}{\pi} \sum_{-\infty}^{+\infty} \frac{\cos s\theta}{z + s}, \qquad 0 < \theta < 2\pi.$$

Combining we have

$$\sin (x + \pi z - \theta z) = \frac{\sin \pi z}{\pi} \sum_{-\infty}^{+\infty} \frac{\sin (x + s\theta)}{z + s}, \qquad 0 < \theta < 2\pi. \tag{2}$$

Denote the function of θ which is the sum of the series on the right by $f(\theta)$. Now $f(\theta)$ is periodic with period 2π. Denote $\theta - 2n\pi$ by $\bar{\theta}$. Then if θ lies in the interval

$$2n\pi < \theta < 2(n + 1)\pi, \tag{3}$$

n a positive integer or zero, $\bar{\theta}$ lies between 0 and 2π and accordingly

$$\sin (x + \pi z - \bar{\theta} z) = \frac{\sin \pi z}{\pi} \sum_{-\infty}^{+\infty} \frac{\sin (x + s\theta)}{z + s}, \qquad \begin{matrix} 0 < \bar{\theta} < 2\pi, \\ 2n\pi < \theta < 2(n + 1)\pi. \end{matrix}$$

Hence for θ in the interval (3),

$$\sum_{-\infty}^{+\infty} \frac{\sin (x + s\theta)}{z + s} = \frac{\pi}{\sin \pi z} \sin \left[(2n + 1)\pi z + x - \theta z\right]. \tag{4}$$

Now divide by a, set $\theta = a$ and $z = x/a$. We get

$$F(x, a) = \frac{\pi}{a} \frac{\sin \left[(2n + 1)\pi \dfrac{x}{a}\right]}{\sin \dfrac{\pi x}{a}}, \qquad 2n\pi < a < 2(n + 1)\pi, \tag{5}$$

where n is a positive integer or zero.

The series (2) being a Fourier's series, we have for $\theta = 2n\pi$,

$$(6)\quad f(2n\pi) = f(0) = \frac{\pi}{2\sin \pi z}[\sin(x + \pi z) + \sin(x - \pi z)] = \pi \sin x \cot \pi z.$$

Hence

$$(7)\qquad F(x, a) = \frac{\pi}{a} \sin x \cdot \cot\left(\pi \frac{x}{a}\right), \qquad a = 2n\pi, \qquad n \geqq 1.$$

Formulae (6) and (7) indicate that $F(x, a)$ has no value for $a = 0$. Clearly from the definition of $F(x, a)$ it follows that

$$F(x, -a) = F(x, a).$$

Thus $F(x, a)$ is evaluated for all real values of x and a for which it is defined.

The behavior of the function at the discontinuities $a = 2n\pi$ is best brought out by expressing formulas (5) and (7) in a different form. Using the trigonometric identity

$$\tfrac{1}{2} + \sum_{m=1}^{n} \cos m\theta = \frac{\sin\left(\frac{2n+1}{2}\theta\right)}{2\sin\frac{\theta}{2}},$$

one obtains

$$(8)\quad F(x, a) = \frac{2\pi}{a}\left(\tfrac{1}{2} + \sum_{m=1}^{n} \cos m\frac{2\pi x}{a}\right), \qquad 2n\pi < a < 2(n+1)\pi, \qquad n \geqq 0,$$

$$(9)\quad F(x, a) = \frac{2\pi}{a}\left(\tfrac{1}{2} + \tfrac{1}{2}\cos x + \sum_{m=1}^{n-1} \cos m\frac{2\pi x}{a}\right), \qquad a = 2n\pi, \qquad n \geqq 1.$$

These last formulae bring out clearly the behavior of the function in the neighborhood of points $a = 2n\pi$, $n \geqq 1$.

3580. [1932, 607]. *Proposed by J. B. Reynolds, Lehigh University.*

If a body dropping vertically from an airplane reaches a velocity of 120 mi./hr., which is 99 percent of its terminal velocity, in 40 sec. what is the value of n on the assumption that the resistance of the air is directly proportional to the weight of the body and the nth power of its velocity?

Solution by Elijah Swift, University of Vermont.

I assume that by "terminal velocity" is meant the limiting velocity approached as the time becomes infinite.

By hypothesis, the force on the body at any time is made up of gravity, downwards, and a resisting force, proportional to mass times the nth power of the velocity. Let x be the distance from the starting point, measured downwards as positive, v the velocity, t the time in seconds, k the constant of proportionality, g the acceleration due to gravity. Then the equation of motion is

$$m\frac{d^2x}{dt^2} = m\frac{dv}{dt} = mg - kmv^n.$$

The terminal velocity, v_T, is given by the equation $g - kv_T^n = 0$. Separating the variables and integrating, we have

$$\int_0^{.99V_T} \frac{dv}{g - kv^n} = \int_0^{40} dt.$$

The substitution $v = v_T z$ reduces this to the equation

$$\int_0^{.99} \frac{dz}{1 - z^n} = \frac{40g}{v_T},$$

since $kv_T^n = g$; or, reducing 120 mi./hr. to ft./sec. and substituting in the given values ($g = 32$), we have to solve for n the equation

$$I_n = \int_0^{.99} \frac{dz}{1 - z^n} = 7.2.$$

I_n clearly decreases as n increases. We compute its value for several values of n, obtaining the following table

$n =$	1,	$\frac{1}{2}$,	$\frac{2}{3}$
$I_n =$	4.605,	8.606,	6.603
$I_n - 7.2 =$	$-$ 2.595,	$+$ 1.406,	$-$ 0.597.

Interpolation or a rough graph shows that $n = 0.6$ is close to the correct value of n. Computing $I_{0.6}$, we find $I_{0.6} = 7.275$ (the last figure inaccurate) and interpolating we find $n = 0.607$ as a more accurate value. The value 0.6 is, however, accurate to within 1 percent and gives a value to I_n accurate to the same percent. Considering the nature of the data, the value $n = 0.6$ may be taken as the correct value of n and as verified by direct computation.

3581. [1932, 607]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Let a, b, c, be the edges of a trihedral angle S-abc, and a', b', c', the perpendiculars at S to the planes bc, ca, ab. If u is the axis of a cone of revolution passing through a, b, c, the line u is also the axis of a cone of revolution tangent to the planes $a'b'$, $b'c'$, $c'a'$.

Solution by S. Vatriquant, Athénée Royale d'Ixelles, Brussels (Belgium).

Cutting the figure by a sphere of centre S, we obtain two spherical triangles ABC and $A'B'C'$, which are reciprocally polar, i.e. such that each vertex of one of them is the pole of the opposite side in the other. The cone passing through a, b, c, cuts the sphere along the circumscribed circle of ABC, and the line u meets the sphere at the pole U of this circle. If we join on the sphere (by arcs of

great circles) AU, BU, CU, these lines are equal and cut the sides of $A'B'C'$, respectively, at M, N, P, so that we have $AM=BN=CP=90°$. Thus $MU=NU=PU$, and since arcs issued from a pole are perpendicular to the polar circles, U is equidistant from the sides of $A'B'C'$. Hence U is the pole of the inscribed circle, cut on the sphere by the cone tangent to the planes $a'b'$, $b'c'$, $c'a'$, and the line u is the axis of this cone.

Solved also by A. D. Bradley, A. Pelletier, and W. H. Rasche.

3583. [1932, 608]. *Proposed by H. Grossman, New York.*

If from any point on a circle, chords be drawn to all vertices of a regular inscribed $3n$-gon, the sum of the n longest chords is equal to the sum of the $2n$ shortest ones.

Solution by Laurence Hadley, Purdue University.

Ptolemy's Theorem may be stated thus: The product of the diagonals of a convex quadrilateral inscribed in a circle equals the sum of the products of the opposite sides. Let ABC be an equilateral triangle, and let P be any point on the arc CA of the circumscribed circle. Applying Ptolemy's theorem to $ABCP$, we have $PB\cdot AC=PA\cdot BC+PC\cdot AB$. Since $AC=BC=AB$, we have at once $PB=PA+PC$. Therefore the following statement may be made as a corollary under Ptolemy's Theorem:

If chords be drawn from any point on a circle to the vertices of an inscribed equilateral triangle, the longest chord, so determined, equals the sum of the other two chords.

To solve the problem proposed, let us assume as given a regular inscribed $3n$-gon with P as any point on the circumscribing circle. Letter the vertices

$$A_1, A_2, \cdots A_n\,; \qquad B_1, B_2, \cdots, B_n\,; \qquad C_1, C_2, \cdots, C_n$$

in counter clock-wise direction beginning with $A_1=A$, where P lies on the arc C_nA_1. Denote the chords from P to the vertices of the polygon as follows:

$$PA_i = a_i\,; \qquad PB_i = b_i\,; \qquad PC_i = c_i\,; \qquad i = 1, 2, \cdots, n.$$

It is clear that $A_iB_iC_i$ is an equilateral triangle. If P is, for example, at the end A_1 of arc C_nA_1 then we may take the b's as the longest chords and the a's and c's as the shortest chords. In this case the shortest of the first set is equal in length to the longest of the second set, i.e., $b_1=c_1=$ a side of equilateral triangle $A_1B_1C_1$, and $a_1=0$. If P is not at an end of C_nA_1, the b chords are each longer than a side of the inscribed equilateral triangle while the a and c chords are each shorter than a side of such a triangle.

Applying the above corollary to the equilateral triangles $A_iB_iC_i$, we have

$$b_i = a_i + c_i, \qquad \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} c_i.$$

The last equation gives the desired result.

Solved also by M. G. Boyce, A. D. Bradley, W. B. Campbell, Mannis Charosh, D. C. Duncan, H. G. Green, A. S. Householder, Frank Irwin, A. Pelletier, and F. Underwood.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

AN OFFICIAL EMBLEM OF THE MATHEMATICAL ASSOCIATION

In response to numerous requests at various times by members of the Association, the Trustees have authorized the manufacture and sale of an official emblem to be worn either as a pin or as a button. A description of the emblem may be found on the inserted sheet at the end of the August-September issue of the Monthly. The design of the emblem was taken from the monogram which has been used on official publications of the Association for many years, but this emblem is much more beautiful because the central figure of an icosahedron is shown in bold relief. Of course the purchase and use of the emblem is entirely optional with the member.

The two hundredth anniversary of the discovery of the normal probability curve falls this year. The discovery was made by DeMoivre, who published his results on November 12, 1733, in a brief paper of seven pages which he presented privately to a few friends. This paper was entitled "Approximatio ad summam terminarum Binomii $(a-b)^n$ in Seriem Expansi." Only two copies of this rare paper are reported extant.

The Econometric Society, an international society for the advancement of economic theory in its relation to statistics and mathematics, has established a quarterly journal, entitled *Econometrica*, under the editorship of Ragnar Frisch, with H. T. Davis, A. H. Hansen, and F. C. Mills as associate editors. The editorial advisory board includes Irving Fisher, Harold Hotelling, Oystein Ore, and C. F. Roos. The business address of the journal is care of Alfred Cowles, Mining Exchange Building, Colorado Springs, Colo.

The degree of Doctor of Science was conferred upon Lipót Fejér, professor of mathematics at the University of Budapest, by Brown University at its commencement exercises on June 16.

The degree of Doctor of Science was conferred upon E. W. Brown, Josiah Willard Gibbs professor of mathematics, emeritus, of Yale University, by that institution on June 16. The following are some statements concerning Professor Brown made by Professor William Lyon Phelps at the conferring of the degree: Professor Brown was born at Hull, England, and received his B. A. degree at Christ College, Cambridge, in 1887. In 1891 he became professor of mathematics at Haverford College where he remained until he was called to Yale in

1907. He became professor emeritus in 1932. His publications on lunar theory and celestial mechanics have given him an international reputation. His tables on the motion of the moon is a monumental work. In youth he expected to be concert pianist, but later took up the music of the spheres.

At the commencement exercises at Harvard University the degree of Doctor of Science was conferred on Professor G. D. Birkhoff. Professor Birkhoff has also been elected Perkins professor of mathematics to succeed Professor W. F. Osgood, who on September 1 retired with the title of professor emeritus.

Oglethorpe University has conferred an honorary doctorate on Professor Archibald Henderson of the University of North Carolina.

Professor J. F. Reilly of the State University of Iowa has been elected chairman of the mathematics section of the Iowa Academy of Science, year 1933–34.

D. E. Smith, professor emeritus of Teachers College, Columbia University, has been decorated by Persia for his study of the Mathematical Works and Philosophy of Omar Khayyam.

The following will work at the School of Mathematics in the Institute for Advanced Study at Princeton: A. A. Albert, University of Chicago; Meyer Salkover, University of Cincinnati; T. Y. Thomas, Princeton University; E. R. vanKampen, Johns Hopkins University; W. E. Bleick, Johns Hopkins University.

Dr. J. V. Atanasoff of Iowa State College, has been promoted to an assistant professorship.

Associate Professor E. T. Browne, of the University of North Carolina has been promoted to a professorship of mathematics.

Dr. L. P. Eisenhart, professor of mathematics and dean of the faculty at Princeton University, has been elected dean of the graduate school.

Assistant Professor W. H. Gage, of Victoria College, has been appointed assistant professor of mathematics at the University of British Columbia.

Alan Hazeltine has returned to the Stevens Institute of Technology as professor of physical mathematics.

Dr. I. O. Horsfall, of Cornell University, has been appointed president of Snow College, Ephraim, Utah.

Assistant Professor K. W. Lamson of Lehigh University has been promoted to an associate professorship of mathematics.

Assistant Professor A. E. Meder of the New Jersey College for Women has been promoted to an associate professorship of mathematics.

Professor Helen Merrill of Wellesley College has retired as professor emeritus.

Dr. F. H. Miller of Columbia University has been appointed to an assistant professorship at Cooper Union.

Professor B. P. Reinsch of the Southern Methodist University, Dallas, Texas, has been appointed professor and head of the department of mathematics at Southern College, Lakeland, Florida.

Assistant Professor C. N. Shuster of the State Teachers College, Trenton, New Jersey, has been promoted to a professorship.

Professor Mary C. Spencer of Newcomb College has retired as professor emeritus.

Professor W. W. Weber of Columbia College, South Carolina, has been appointed professor of mathematics at Lander College.

G. A. Yanosik has been promoted to an assistant professorship at New York University.

Dr. A. L. Foster has been appointed instructor at the University of California at Berkeley.

Miss Nancy Cole has been appointed instructor at Sweet Briar College.

Miss Vevia Blair, Head of the department of mathematics at Horace Mann School for Girls, an observation school of Teachers College, Columbia University, died July 9, 1933, at the age of 62.

PROBABILITY FUNCTIONS AND STATISTICAL PARAMETERS

By L. S. HILL, Hunter College of the City of New York

1. *Preliminary Notes*

It is the purpose of this article to present a connected account of the more important concepts used in the modern theory of statistical distributions of sets of n continuous quantitative attributes. For the effective accomplishment of this purpose, it has seemed wise to avoid, so far as possible, any discussion of the philosophical groundwork, and to devote the space thus saved to a more complete exposition of the working ideas, about which little, if any, controversy exists.

Some form of definition of the fundamental terms is, however, necessary. That given in Sections 2 and 3 has the merit of being brief and explicit, and of affording a geometrical background against which the interrelations of the statistical ideas can be conveniently displayed. The circumstance that its soundness may be questioned in some quarters is, from our present standpoint, of only limited interest; for, as already stated, we are not attempting to deal with "foundations."

Certain notions of the theory of sets of points in an n-dimensional Euclidean space E_n will be required. If ϵ is an arbitrary positive real number, and $a=(a_1, a_2, \cdots, a_n)$ is a point of E_n, we denote by $V_\epsilon(a)$ the set of all those points of E_n of which the distances from a are less than ϵ; thus $V_\epsilon(a)$ consists of all points $x=(x_1, x_2, \cdots, x_n)$ of E_n such that $\sum_{j=1}^{n}(x_j-a_j)^2<\epsilon^2$.

A set J of points of E_n constitutes a *region* in E_n if, for each point a of J, there exists an $\epsilon>0$ such that every point of $V_\epsilon(a)$ is a point of J. The *frontier* of a region J consists of all those points of E_n which, while not points of J, are limit points of J; and the frontier is a closed set (a set containing all its limit points). The term *range* will be applied to any set in E_n which is either (1) a region, or (2) a region supplemented by the adjunction of some, or all, points of its frontier. The space E_n is a region.

If, for some point a of E_n, there exists a positive real number c (sufficiently great) such that every point of the range R is a point of the region $V_c(a)$, then R is called a *bounded* range. Let R be a bounded range in E_n, and let a, b denote points of R. Then the least upper bound of the distance (a, b), as a and b vary independently over R, is called the *diameter* of R.

Every bounded range R in E_n has a definite (positive) Lebesgue measure, which we shall call the *extent* of R. Extent coincides with length, area, volume, $\cdots$, in the cases $n=1, 2, 3, \cdots$.

2. *Statistical Populations, and Typical Samples*

Let P denote any set for each element (individual) of which n characteristics (attributes) are distinguished. Let each of these attributes be measurable